Finite hyper-elastodynamics

$$(S) \Leftrightarrow (W) \approx (G) \Leftrightarrow (M)$$

ODE's

$$\sigma_{ij,j} + \rho b_i = 0$$

D'Alembert principle. $b \leftarrow b - a$

$$\int_{v} w_i \rho a_i + w_{(i,j)} \sigma_{ij} \, dx = \int_{v} w_i b_i \rho \, dx + \int_{a_h} w_i h_i \, da$$

pull-back

$$\int_{V} \rho_0 W_i A_i \, dX$$

$$A(X,t) = \dot{V}(X,t) = \ddot{U}(X,t)$$

$$\parallel$$

$$\sum_{A} \sum_{B} \int_{V} \rho_0 N_A(X) \, c_{iA} \; N_B(X) \, \ddot{d}_{iB} \, dX \qquad U_i^h(X,t) = \sum_{B} N_B(X) \, d_{iB}(t)$$

$$\parallel$$

$$\sum_{A,B} c_{iA} \underbrace{\int_{V} \rho_0 N_A(X) N_B(X) \, dX \; \delta_{ij}}_{M_{iAjB}} \; \ddot{d}_{jB}(t)$$

$$M_{iAjB} = M_{PQ} \quad \text{mass matrix}$$

Remark 1: $M_{iAjB} = M_{jBiA} \Leftrightarrow M_{PQ} = M_{QP}$

M is a constant matrix (independent of time)

M is positive-definite. :

$$C^T M C = \sum_A \sum_B C_{iA} \delta_{ij} \int_V \rho_0 N_A N_B \, dX \, C_{jB}$$

$$= \int_V W_i^h W_i^h \rho_0 \, dX \geq 0$$

and $C^T M C = 0$ implies $W_i^h = 0 \Rightarrow C = 0$.

M has the same banded structure as K.

The ODE problem :

$$M \ddot{d}(t) + N(d(t)) = F(t)$$

$$d(0) = d_0$$

$$\dot{d}(0) = V_0$$

given initial data projected to the discrete space

<u>The generalized - α method</u>

$d_n \approx d(t_n)$ $\quad v_n \approx v(t_n)$ $\quad a_n \approx a(t_n)$

determine d_{n+1}, v_{n+1}, a_{n+1}, s.t.

$$M a_{n+\alpha_m} + N(d_{n+\alpha_f}) = F_{n+\alpha_f} = F(t_{n+\alpha_f})$$

$$d_{n+\alpha_f} = (1-\alpha_f) d_n + \alpha_f d_{n+1}$$

$$V_{n+\alpha_f} = (1-\alpha_f) V_n + \alpha_f V_{n+1}$$

$$a_{n+\alpha_m} = (1-\alpha_m) a_n + \alpha_m V_{n+1}$$

$$d_{n+1} = d_n + \Delta t\, V_n + \frac{\Delta t^2}{2}\left[(1-2\beta)\, a_n + 2\beta\, a_{n+1}\right]$$
$$V_{n+1} = V_n + \Delta t\left[(1-\gamma)\, a_n + \gamma\, a_{n+1}\right]$$

4 parameters define the algorithm : $\alpha_f, \alpha_m, \beta, \gamma$.

Remark 1 : We need a_0 to start the algorithm, and

$$M a_0 = F(t_0) - N(d_0).$$

Remark 2 : For linear elastodynamics, analysis results have been pursued by Chung & Hulbert (Journal of Applied Mechanics, 1993, Vol 60, pp. 371-375).

- Second-order time accuracy is achieved if
$$\gamma = \frac{1}{2} - \alpha_f + \alpha_m, \quad \beta = \frac{1}{4}(1 - \alpha_f + \alpha_m)^2.$$
- Unconditional stability:
$$\alpha_m \geq \alpha_f \geq 1/2.$$
- High frequency damping
$$\alpha_m = \frac{2 - \rho_\infty}{1 + \rho_\infty} \qquad \alpha_f = \frac{1}{1 + \rho_\infty}$$

$$\ddot{u} + \underbrace{2\xi\omega}_{\text{damping ratio}}\dot{u} + \underbrace{\omega^2}_{\text{undamped frequency}} u = f \xleftarrow{(K-\omega^2 M)\psi = 0} M\ddot{d} + C\dot{d} + Kd = F$$

$\xi = \left(\frac{a}{\omega} + b\omega\right)/2$ viscous damping $aM + bK$.

discretize $\rightarrow$ $X_{n+1} = A X_n \qquad X_n = \{d_n, \Delta t\, V_n, \Delta t^2 a_n\}^T.$

$\rho = \max(|\lambda_1|, |\lambda_2|, |\lambda_3|)$ spectral radius.

and ρ is a function of $\omega \Delta t$.

Implementation: Given d_n, v_n, a_n

predictor: $v_{n+1}^{(0)} = v_n$

$$a_{n+1}^{(0)} = \frac{\gamma - 1}{\gamma} a_n$$

$$d_{n+1}^{(0)} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2}\left((1-2\beta) a_n + 2\beta a_{n+1}^{(0)} \right)$$

multi-corrector: repeat the following steps for $i = \overset{0}{1}, 2, \cdots, i_{max}$

- $d_{n+\alpha_f}^{(i)} = (1-\alpha_f) d_n + \alpha_f d_{n+1}^{(i)}$

evaluate at intermediate steps. $v_{n+\alpha_f}^{(i)} = (1-\alpha_f) v_n + \alpha_f v_{n+1}^{(i)}$

$$a_{n+\alpha_m}^{(i)} = (1-\alpha_m) a_n + \alpha_m a_{n+1}^{(i)}$$

- Form the residual at intermediate solutions

$$R_{n+1}^{(i)} := -M a_{n+\alpha_m}^{(i)} - N(d_{n+\alpha_f}^{(i)}) + F_{n+\alpha_f}.$$

and tangent ~~vector~~ matrix

$$DR_{n+1}^{(i)} = \frac{\partial R}{\partial a_{n+\alpha_m}} \underbrace{\frac{\partial a_{n+\alpha_m}}{\partial a_{n+1}}}_{\to\, \alpha_m}$$

$$+ \frac{\partial R}{\partial v_{n+\alpha_f}} \underbrace{\frac{\partial v_{n+\alpha_f}}{\partial v_{n+1}}}_{\to\, \alpha_f} \underbrace{\frac{\partial v_{n+1}}{\partial a_{n+1}}}_{\to\, \gamma \Delta t}$$

$$+ \frac{\partial R}{\partial d_{n+\alpha_f}} \underbrace{\frac{\partial d_{n+\alpha_f}}{\partial d_{n+1}}}_{\to\, \alpha_f} \underbrace{\frac{\partial d_{n+1}}{\partial a_{n+1}}}_{\to\, \beta \Delta t^2}.$$

$$DR_{n+1}^{(i)} = -\alpha_m M + \alpha_f \gamma \Delta t \frac{\partial R}{\partial v_{n+\alpha_f}} - \alpha_f \beta \Delta t^2 \, DN(d_{n+\alpha_f}^{(i+1)})$$

- Solve the linear system

$$DR_{n+1}^{(i)} \cdot \Delta a_{n+1}^{(i+1)} = R_{n+1}^{(i)}$$

- update the solution

$$a_{n+1}^{(i+1)} = a_{n+1}^{(i)} + \Delta a_{n+1}^{(i+1)}$$

$$v_{n+1}^{(i+1)} = v_{n+1}^{(i)} + \gamma \Delta t \, \Delta a_{n+1}^{(i+1)}$$

$$d_{n+1}^{(i+1)} = d_{n+1}^{(i)} + \beta \Delta t^2 \, \Delta a_{n+1}^{(i+1)}$$

- Test if $\| R_{n+1}^{(i+1)} \| < tol_A$ or $\| R_{n+1}^{(i+1)} \| < tol_R \| R_{n+1}^{(0)} \|$.

Remark: For linear problems with viscous damping,

$$R_{n+1}^{(i)} = F_{n+\alpha_f} - M a_{n+\alpha_m}^{(i)} - C v_{n+\alpha_f}^{(i)} - K d_{n+\alpha_f}^{(i)}$$

$$DR_{n+1}^{(i)} = -\alpha_m M - \alpha_f \gamma \Delta t \, C - \alpha_f \beta \Delta t^2 K.$$

Remark: $\alpha_m = \alpha_f = 1$: Newmark (should not be used)

$\alpha_m = 1$: HHT-α

Remark: undamped eigenproblem

$$(K - \lambda M)\psi = 0 \qquad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{neq}$$

$$\psi_l^T M \psi_m = \delta_{lm}$$

$$\Rightarrow \quad \{\lambda_l, \psi_l\}_{l=1}^{n_{eq}} \qquad \omega_l = (\lambda_l)^{1/2} \quad \{\psi_l\} \text{ basis of } \mathbb{R}^{n_{eq}}.$$

$$0 = \psi_m^T \left(M \sum_{n=1}^{n_{eq}} \psi_n \ddot{d}_n(t) + C \sum_{n=1}^{n_{eq}} \psi_n \dot{d}_n(t) + K \sum_{n=1}^{n_{eq}} \psi_n d_n(t) - F \right)$$

$$= \ddot{d}_m + 2\xi\omega \dot{d} + \omega^2 d - F_m \qquad \underline{\text{SDOF problem}}$$

$$\psi_m^T M \psi_n = \delta_{mn} \qquad \psi_m^T K \psi_n = \lambda_n \delta_{mn}$$

foundation of numerical analysis for linear dynamics

$$C = aM + bK \qquad \text{Rayleigh damping}$$

$$\psi_m^T C \psi_n = (a + b\lambda_n)\, \delta_{mn}.$$

Hughes FEM book Chap. 9.

Energy analysis of linear problems

Consider a physical problem discretized by the mid-point rule : $\rho_\infty = 1$ (i.e. no high frequency damping)

$$M a_{n+\frac{1}{2}} + C v_{n+\frac{1}{2}} + K d_{n+\frac{1}{2}} = F_{n+1/2}$$

$$(\cdot)_{n+\frac{1}{2}} = \frac{1}{2}(\cdot)_n + \frac{1}{2}(\cdot)_{n+1}$$

$$v_{n+1} = v_n + \frac{\Delta t}{2}(a_n + a_{n+1}) = v_n + \Delta t\, a_{n+1/2}$$

$$d_{n+1} = d_n + \Delta t\, v_n + \frac{\Delta t^2}{2} a_{n+\frac{1}{2}}$$

$$v_{n+\frac{1}{2}} \cdot M a_{n+\frac{1}{2}} = v_{n+\frac{1}{2}} \cdot M \frac{v_{n+1} - v_n}{\Delta t}$$

$$= \frac{1}{2\Delta t}\left(v_{n+1} \cdot M v_{n+1} - v_n \cdot M v_n \right)$$

$$= \frac{1}{\Delta t}\left(K_{n+1} - K_n \right) \qquad K(v_{n+1})$$

$v_{n+\frac{1}{2}} \cdot C v_{n+\frac{1}{2}} := \mathcal{D} \geqslant 0$ dissipation from viscous damping

$$v_{n+\frac{1}{2}} \cdot K d_{n+\frac{1}{2}} = \frac{d_{n+1} - d_n}{\Delta t} \cdot K d_{n+\frac{1}{2}}$$

$$= \frac{1}{2\Delta t}\left(d_{n+1} \cdot K d_{n+1} - d_n \cdot K d_n \right)$$

$$= \frac{1}{\Delta t} \left(U(d_{n+1}) - U(d_n) \right)$$

$$\Rightarrow \quad K(v_{n+1}) + U(d_{n+1}) = K(v_n) + U(d_n) - \Delta t\, \mathcal{D}$$

$\Rightarrow$ v_{n+1} & d_{n+1} are bounded. $\Rightarrow$ stability.

If we return to nonlinear problems: discretized by mid-point:

$$\int_V \rho_0 W_i A_i \, dX + \int_V W_{i,I} P_{iI} \, dX = \int_V W_i \rho_0 B_i \, dX + \int_{A_H} W_i H_i \, dA.$$

$$A_i = \dot{V}_i = \ddot{U}_i \quad \text{discretize} \quad \frac{V_{n+1} - V_n}{\Delta t}$$

- Total linear momentum $L := \int_V \rho_0 V \, dX$

Pick $W = \xi$ constant vector.

$$\Rightarrow \int_V \rho_0 \xi_i \dot{V}_i \, dX + \int_V \underbrace{\xi_{i,I}}_{0} P_{iI} \, dX = \int_V \xi_i \rho_0 B_i \, dX + \int_{A_H} \xi_i H_i \, dA$$

$$\Rightarrow \xi_i \left\{ \frac{D}{Dt} \int_V \rho_0 V_i \, dX - \int_V \rho_0 B_i \, dX - \int_{A_H} H_i \, dA \right\} = 0$$

$$\Rightarrow \boxed{\frac{D}{Dt} L = \int_V \rho_0 B \, dX + \int_{A_H} H \, dA}$$

Conservation of linear mom.

Conditional in the sense that constant vectors have to be admissible test function.

no essential BC.

$$\frac{L_{n+1} - L_n}{\Delta t} = \int_V \rho_0 B_{n+1/2} \, dX + \int_V H_{n+1/2} \, dA.$$

- Total angular momentum.
$$J = \int_V \rho_0 \varphi_t \times V \, dX$$
$$J_i = \int_V \rho_0 \, \varepsilon_{ijk} \varphi_j V_k \, dX.$$

Pick $W = \xi \times \varphi_{n+1/2}$.

$$\int_V \rho_0 W_i A_i \, dX = \int_V \rho_0 \underbrace{\varepsilon_{ijk}}_{= \varepsilon_{jki}} \xi_j x_{k_{n+1/2}} \frac{V_{n+1\, i} - V_{n\, i}}{\Delta t} dX$$

$$= \xi_j \int_V \rho_0 \, \varepsilon_{jki} x_{k_{n+1/2}} \frac{V_{n+1\, i} - V_{n\, i}}{\Delta t} dX$$

$$= \xi \cdot (J_{n+1} - J_n)/\Delta t.$$

Similarily,

$$\int_V \rho_0 \, \xi \times \varphi_{n+1/2} \, B_{n+1/2} \, dX = \xi \cdot \int_V \varphi_{n+1/2} \times B_{n+1/2} \, dX$$

$$\int_{A_H} \xi \times \varphi_{n+1/2} \, H_{n+1/2} \, dA = \xi \cdot \int_{A_H} \varphi_{n+1/2} \times H_{n+1/2} \, dA.$$

Skew tensor and its axial vector.

$$W_\xi u = \xi \times u$$

$\rightarrow$ skew tensor : $W_{ij} = -\varepsilon_{ijk} \xi_k$

axial/dual vector: $\xi_k = -\frac{1}{2} \varepsilon_{ijk} W_{ij}$

$$\int_V (\xi \times \varphi_{n+1/2\, i})_{,I} P_{iI}\, dX = \int_V W_{\xi\, ij}\, \varphi_{n+1/2\, j,I} P_{iI}\, dX$$

$$= \int_V W_{\xi\, ij}\, F_{jI} P_{iI}\, dX$$

$$= \underbrace{W_{\xi\, ij}}_{\text{skew}} \underbrace{\int_V F_{iJ} S_{JI} F_{jI}\, dX}_{\text{symm.}}$$

$$= 0$$

$$\Rightarrow \quad \xi \cdot \left\{ \frac{J_{n+1} - J_n}{\Delta t} - \int_V \varphi_{n+1/2} \times B_{n+1/2}\, dX - \int_{A_H} \varphi_{n+1/2} \times H_{n+1/2}\, dA \right\}$$

$$= 0$$

$$\Rightarrow \quad \boxed{\frac{J_{n+1} - J_n}{\Delta t} = \int_V \varphi_{n+1/2} \times B_{n+1/2}\, dX + \int_{A_H} \varphi_{n+1/2} \times H_{n+1/2}\, dA}$$

resultant moment.

1) $\xi \times X$ needs to be admissible.

2) Stress needs to be symmetric.

• Total energy (Hamiltonian)

Pick $W_i = V_{i\, n+1/2}$ ← G data is time independent.

$$\mathcal{S}_U = \{ U_h : \dots \quad U_h(\cdot, t) = G \text{ on } A_G \}$$
$$\mathcal{S}_V = \{ V_h : \dots \quad V_h(\cdot, t) = \dot{G} \text{ on } A_G \}$$

$$\int_V \overset{①}{\rho_0 V_{i\, n+1/2} (V_{n+1} - V_n)/\Delta t} + \overset{②}{V_{i\, n+1/2, I}\, P_{iI}}\, dX$$

$$= \int_V V_i \rho_0 B_i \, dX + \int_{A_H} V_i H_i \, dA$$

$$① = \int_V \frac{\rho_0}{2\Delta t} \left(|V_{n+1}|^2 - |V_n|^2 \right) dX$$

$$② = \int_V V_{i\, n+1/2, I} \; F_{iJ\, n+1/2}\, S_{JI} \, dX$$

$$V_{i\, n+1/2} = \frac{U_{i\, n+1} - U_{i\, n}}{\Delta t}.$$

$$V_{i\, n+1/2, I} = \frac{1}{\Delta t} \left(F_{iI\, n+1} - F_{iI\, n} \right).$$

$$F_{iJ\, n+1/2}\, V_{i\, n+1/2, I} = \frac{1}{2\Delta t} \left(C_{n+1} - C_n \right)_{IJ}$$

$$② = \int_V \frac{1}{2\Delta t} \left(C_{n+1} - C_n \right) : S \, dX.$$

In general. $\frac{1}{2}(C_{n+1} - C_n) : S_{n+1/2} \neq \Phi(C_{n+1}) - \Phi(C_n)$

Simo : Determine a collocation parameter θ such that

$$\frac{1}{2}(C_{n+1} - C_n) : S(C_{n+\theta}) = \Phi(C_{n+1}) - \Phi(C_n)$$

and θ is ~~d~~ solved from the above eqn. at each quadrature pt.

See, Simo & Tarnow ZAMP 1992, 43: 757-792.

Laursen & Meng, CMAME 2001, 190: 6309-6322.

Gonzalez. discrete gradient CMAME. 2000, 190: 1763-1783.

$$S_{alg} := S(C_{n+1/2}) + \frac{\phi(C_{n+1}) - \phi(C_n) - S(C_{n+1/2}) : Z_n}{\| Z_n \|^2} Z_n$$

$$Z_n := (C_{n+1} - C_n)/2.$$

$$\boxed{Z_n : S_{alg} = \phi(C_{n+1}) - \phi(C_n)}$$

directionality property

- enhancement.
- $O(\|Z_n\|^2)$ do not destroy the temporal accuracy.

S_{alg} is symm.

angular mom. conservation.

Replace S by S_{alg} in the ~~momentum~~ weak-form problem, one has

$$\boxed{\begin{aligned} &\frac{1}{\Delta t}\int_{V} \frac{\rho_0}{2}|V_{n+1}|^2 + \Phi(C_{n+1})\,dX \\ &- \frac{1}{\Delta t}\int_{V} \frac{\rho_0}{2}|V_n|^2 + \Phi(C_n)\,dX \\ &= \int_{V} V_{n+1/2}\cdot \rho_0 B\,dX + \int_{A_H} V_{n+1/2}\cdot H\,dA \end{aligned}}$$

Energy preservation / stability.

Remark: High-mode dissipation can be added. See, e.g. Armero & Romero, CMAME. 2001. 190: 2603-2649.