Elastic constitutive relations

- Objectivity

Consider a superimposed rigid body motion:

$$x \mapsto x^+ = \underset{\substack{\cap \\ SO(3)}}{Q}\, x + \underset{\substack{\cap \\ \mathbb{R}^3}}{c}$$

proper orthogonal group.

rigid means
$$x_1^+ - x_2^+ = Q\,[x_1 - x_2]$$
$$\Rightarrow \quad \|x_1^+ - x_2^+\| = \|x_1 - x_2\|$$

Euclidean distance
$$\|\cdot\| := (\cdot)^T(\cdot)$$

$$F := \frac{\partial x}{\partial X} \qquad F^+ := \frac{\partial x^+}{\partial X} = \frac{\partial x^+}{\partial x}\frac{\partial x}{\partial X} = QF.$$

or
$$F^+_{aA} = Q_{ai}F_{iA}$$

A spatial tensor/vector/scalar is said to transform <u>objectively</u> if they transform under standard rules of tensor analysis:

$$A^+(x^+, t^+) = Q(t)\,A(x,t)\,Q^T(t)$$

rank-2 tensor

$$u^+(x^+, t^+) = Q(t)\,u(x,t)$$

vector

$$\rho^+(x^+, t^+) = \rho(x, t)$$

scalar.

examples: • $J = \det F$. $\qquad J^+ = \det F^+ = \det Q \det F = J$

$\Rightarrow$ scalar field J is objective

• $l = \dot{F} F^{-1} \longrightarrow l^+ = \dot{F}^+ F^{+^{-1}}$

$$= \dot{\overline{(QF)}}(QF)^{-1}$$

l is not suitable for constitutive relations.

$$= \dot{Q}Q + Q\dot{F}F^{-1}Q^T$$

l does NOT transform objectively. $\Leftarrow$

$$= \underbrace{\dot{Q}Q}_{\text{skew tensor}} + Q l Q^T$$

• $l = d + \omega$

$d^+ = Q d Q^T \Rightarrow$ rate-of-strain is objective.

$\omega^+ = Q \omega Q^T + \dot{Q}Q$

• $t = \sigma n \qquad t^+ = \sigma^+ n^+$

$t^+ = Qt \qquad n^+ = Qn$

$\Rightarrow \quad Q\sigma n = \sigma^+ Q n$

$\Rightarrow \quad \sigma^+ = Q \sigma Q^+$

Cauchy stress is objective.

Remark: the material time derivative of σ is NOT objective.

$$\frac{D}{Dt}\sigma = \left\{\frac{D}{Dt}\sigma(\varphi(X,t),t)\right\}\circ\varphi_t^{-1}$$

$$= \frac{\partial}{\partial t}(\sigma_t\circ\varphi_t)\circ\varphi_t^{-1}$$

$$= \frac{\partial}{\partial t}\sigma_t + \nabla\sigma_t\, v_t$$

$$\sigma_t^+ = Q(t)\,\sigma_t\,Q^T(t)$$

$$\dot{\sigma}_t^+ = \dot{Q}_t\sigma_t Q_t^T + Q_t\dot{\sigma}_t Q_t^T + Q_t\sigma_t\dot{Q}_t^T$$

$$= Q(t)\,\dot{\sigma}_t\,Q^T(t) + [\dot{Q}Q^T]\sigma_t^+ + \sigma_t^+[\dot{Q}Q^T].$$

There are several ways for modifying the stress rate definition, and they are known as the objective stress rates.

- Frame indifference.

$$P(X,t) = \frac{\partial\Phi(X,F(X,t))}{\partial F}$$

elastic material stress depends on the current def. state

We demand the potential energy to remain invariant under super-imposed rigid motions, i.e.,

$$\Phi(X,F) = \Phi(X,QF) \quad \text{for all } Q\in SO(3)$$

Recall that $F = RU$, picking $Q = R^T$.

$$\bar{\Phi}(X, F) = \bar{\Phi}(X, U) \underset{\uparrow}{=} \hat{\Phi}(X, C)$$

$$C = U^2$$

→ for elastic materials, the energy depends on the deformation state through U, or equivalently, C.

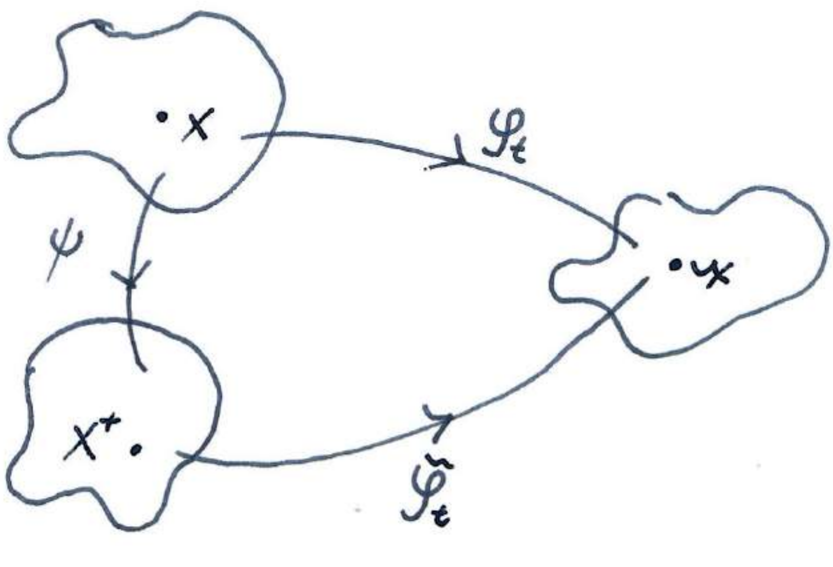
→ $\hat{\Phi}(X, C)$ is objective since $C^+ = C$.

In your material model routine, it is a good option by writing functions with C as an input, rather than F.

- Isotropy

Let X be a material point in the referential configuration.
If a rigid deformation is imposed on the referential configuration,

$$\psi(X) = X^+ = QX + c. \qquad Q \in SO(3)$$



$$\underline{\underline{\varphi_t = \tilde{\varphi}_t \circ \psi}}$$

$$F_t = \frac{\partial \varphi_t}{\partial X} = \frac{\partial \tilde{\varphi}_t}{\partial X^+}\frac{\partial X^+}{\partial X} = F_t^+ Q \Rightarrow F_t^+ = F_t Q^T$$

$$\Rightarrow C^+ = Q C Q^T$$

In general $\phi(C) \neq \phi(C^+)$

$$G_X := \{ Q \in SO(3) : \bar{\Phi}(X, QCQ^T) = \bar{\Phi}(X, C) \}$$

is a subgroup of $SO(3)$ at point X, and if $G_X = SO(3)$, the material is <u>isotropic</u>; otherwise it is <u>anisotropic</u>.

Representation theorem: A function f of symmetric tensors is isotropic if and only if

$$f(H) = f(QHQ^T) \quad \text{for all } Q \in SO(3).$$

An isotropic function depends on H through its principal invariants: $I_1 = \operatorname{tr} H$

$$I_2 = \tfrac{1}{2}(I_1^2 - \operatorname{tr} H^2)$$

$$I_3 = \det H.$$

For isotropic elastic materials, one may write the stored energy as $\phi(C) = \tilde{\Phi}(I_1(C), I_2(C), I_3(C))$.

Remark: $\Phi(RCR^T) = \Phi(C)$

$$\begin{aligned}\Phi(RCR^T) &= \Phi(C)\\ &\parallel\\ \Phi(RUU^TR^T) &= \Phi(FF^T) = \Phi(b)\end{aligned}$$

Only for isotropic elastic response, the stored energy depends on the motion through b.

- Coleman - Noll procedure.

Let Ω_0 be an arbitrarily chosen region in the ref. configuration; and $\Omega_t = \varphi_t(\Omega_0)$.

$\iota(x, t)$ internal energy per unit volume

$$\frac{D}{Dt}\int_{\Omega_t} \frac{1}{2}\rho|v|^2 + \iota \, dx = \int_{\partial\Omega_t} t\cdot v - q\cdot n \, da + \int_{\Omega_t} \rho b\cdot v + r \, dx.$$

kinetic energy

stress power

true or Cauchy heat flux

power of body force

radiation / heat source

pull the above back to Ω_0:

$\rho(x, t) J(X, t) = \rho_0(X)$. $\quad v_t \circ \varphi_t = V_t$.

introduce $\iota(x, t) J(X, t) = \mathcal{I}(X, t)$

$$t_i \, da = T_i \, dA \Rightarrow \int_{\partial\Omega_t} t\cdot v \, da = \int_{\partial\Omega_0} T\cdot V \, dA$$

We introduce nominal heat flux (or Piola-Kirchhoff heat flux) as Q and

$$\int_{\partial\Omega_0} \underline{Q} \cdot N \, dA := \int_{\partial\Omega_t} q \cdot n \, da$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$$

$$\int_{\partial\Omega_0} \underline{q \cdot JF^{-T}} N \, dA$$

We may conclude that $Q = JF^{-1} q$

$$\hookrightarrow J \chi_x^{-1}(q) \text{ piola transformation.}$$

$$r(x,t) \, J(X,t) := R(X,t)$$

$$\frac{D}{Dt} \int_{\Omega_0} \frac{1}{2} \rho_0 |V|^2 + \mathcal{I} \, dX = \int_{\partial\Omega_0} T \cdot V - Q \cdot N \, dA$$

$$+ \int_{\Omega_0} \rho_0 B \cdot V + R \, dX.$$

Ω_0 is arbitrary, we may localize the above to PDE form:

$$\rho_0 V \cdot \frac{DV}{Dt} + \frac{D\mathcal{I}}{Dt} = \mathrm{DIV}\, P \cdot V + P : \dot{F} - \mathrm{DIV}\, Q + \rho_0 B \cdot V + R.$$

Recall that the momentum eqn. is

$$\mathrm{DIV}\, P + \rho_0 B = 0$$

D' Alembert principle : $b \leftarrow b - a$

$B \leftarrow B - A$

$$\mathrm{DIV}\, P + \rho_0 \left(B - \frac{Dv}{Dt} \right) = 0$$

$$\Rightarrow \quad \mathrm{DIV}\, P \cdot v + \rho_0 v \cdot \left(B - \frac{Dv}{Dt} \right) = 0$$

$$\Leftrightarrow \quad \frac{D}{Dt} \left\{ \frac{\rho_0}{2} |v|^2 \right\} = \mathrm{DIV}\, P \cdot v + \rho_0 B \cdot v$$

Balance of mechanical energy in material description

Plug into the balance of total energy :

$$\frac{D}{Dt} \mathcal{I} = P : \dot{F} - \mathrm{DIV}\, Q + R .$$

Remark : It is often to see people use the internal energy per <u>mass</u> , and there will be ' ρ_0 ' on the left hand side of the above eqn.

Remark : The stress power term can be expressed in various different forms :

$$P : \dot{F} = J \sigma : l = \underline{J \sigma : d} = \underline{\tau : d} = \underline{S : \dot{E}}$$

Mandel stress . $\Sigma := CS$ (used in plastic materials)

$$S : \dot{E} = S : \tfrac{1}{2}\dot{C} = \Sigma : \tfrac{1}{2} C^{-1}\dot{C}$$

Co-Rotated Cauchy stress : $\sigma_u := J^{-1} U S U$

introduced by Green & Naghdi

$$= U F^{-1} \sigma F^{-T} U$$

$$= R^T \sigma R$$

$$J\sigma : d = J R^T \sigma R : \underbrace{R^{-1} d R^{-T}}_{D_R} = J \sigma_u : D_R$$

D_R. rotated rate of deformation.

Biot Stress . $T_B := R^T P = R^T F S = U S$

$$P : \dot{F} = P : [\dot{R}(R^T R) U + R \dot{U}]$$

$$= P F^T : \dot{R} R^T + R^T P : \dot{U}$$

$$= \tau : \dot{R} R^T + T_B : \dot{U}$$

$$= \mathrm{symm}(T_B) : \dot{U} .$$

Work conjugate pairs :

$J\sigma$	P	S	Σ	$J\sigma_u$	$\mathrm{symm}\, T_B$
d	$\dot{F}$	$\dot{E}$	$\tfrac{1}{2} C^{-1}\dot{C}$	D_R	$\dot{U}$.

2nd law:

$$\cancel{\mathcal{D}} := \frac{D}{Dt}\int_{\Omega_0} \eta \, dx + \int_{\partial\Omega_0} \frac{Q}{\Theta}\cdot N \, da - \int_{\Omega_0} \frac{R}{\Theta} \, dx \geq 0$$

localization

positive. absolute temperature

Clausius-Duhem inequality

$$\cancel{\mathcal{D}} := \frac{D\eta}{Dt} - \frac{R}{\Theta} + \underbrace{\mathrm{DIV}\frac{Q}{\Theta}}_{\frac{1}{\Theta}\mathrm{DIV}\,Q - Q\cdot\mathrm{GRAD}\,\Theta\,\frac{1}{\Theta^2}} \geq 0$$

$$Q\cdot\mathrm{GRAD}\,\Theta = J F^{-1} q\cdot\mathrm{GRAD}\,\Theta$$
$$= J\, q\cdot\mathrm{grad}\,\Theta \leq 0$$

physical obervation: heat flux points from high to low temperature.

$$\Rightarrow \quad \Theta\frac{D\eta}{Dt} + \underbrace{\mathrm{DIV}\,Q - R}_{P:\dot{F} - \frac{D}{Dt}\mathcal{I}} \geq 0$$

$$\Rightarrow \quad P:\dot{F} - \frac{D}{Dt}\mathcal{I} + \Theta\frac{D}{Dt}\eta \geq 0$$ Clausius-Plank inequality

Helmholtz free energy $\Phi := \mathcal{I} - \Theta \eta$

$$\Rightarrow \quad \frac{D}{Dt}\Phi = \frac{D}{Dt}\mathcal{I} - \eta \frac{D}{Dt}\Theta - \Theta \frac{D}{Dt}\eta .$$

Clausius - Planck inequality:

$$P : \dot{F} - \dot{\Phi} - \eta \dot{\Theta} \geq 0$$

if we are working on a pure mechanical process:

$$P : \dot{F} - \dot{\Phi} \geq 0$$

If $\Phi = \Phi(F)$, then $\quad P : \dot{F} - \dfrac{\partial \Phi}{\partial F} : \dot{F} \geq 0$

or $\quad \left(P - \dfrac{\partial \Phi}{\partial F} \right) : \dot{F} \geq 0$

We choose $\underline{P = \dfrac{\partial \Phi}{\partial F}}$.

perfectly elastic material: no dissipation / entropy production

Remark: We assume $\Phi(I) = 0$, i.e., strain energy vanishes in the ref. Config. This is known as the <u>normalization condition</u>.

$\Phi(F) \geqslant 0$: the stored energy 'increases' with deformation

~~If~~ ~~the~~ The above two assumptions ensures the stress vanishes in the ref. configuration.

For isotropic materials, due to the representation theorem

$$\phi = \phi\left(I_1(C),\ I_2(C),\ I_3(C) \right)$$

Recall $\quad P : \dot{F} = S : \dot{E} = \frac{1}{2} S : \dot{C}$

$$\frac{1}{2} S : \dot{C} - \frac{\partial \phi}{\partial C} : \dot{C} \geqslant 0 \quad \text{implies} \quad S = 2 \frac{\partial \Phi}{\partial C}$$

$$S = 2 \frac{\partial \Phi}{\partial I_1} \underbrace{\frac{\partial I_1}{\partial C}}_{=\, I} + 2 \frac{\partial \Phi}{\partial I_2} \underbrace{\frac{\partial I_2}{\partial C}}_{=\, I_1 I - C} + 2 \frac{\partial \Phi}{\partial I_3} \underbrace{\frac{\partial I_3}{\partial C}}_{=\, I_3 C^{-1}}$$

$$= 2 \left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2} \right) I - \frac{\partial \Phi}{\partial I_2} C + I_3 \frac{\partial \Phi}{\partial I_3} C^{-1} \right]$$

Constitutive eqn. in principal invariants

$$\Phi = \Phi(\lambda_1, \lambda_2, \lambda_3)$$

$$S = \sum_{a=1}^{3} \frac{1}{\lambda_a} \frac{\partial \Phi}{\partial \lambda_a} N_a \otimes N_a$$

constitutive eqn. in principal stretches

See Holzapfel. pp. 219 – 221.

More refined constitutive theory:

$$\Phi(C) = \Phi_{vol}(J) + \underbrace{\Phi_{ich}}_{\text{isochoric}}(\bar{C}) \qquad \bar{F}^T\bar{F} = J^{-2/3}C$$

$$S = \underbrace{2\frac{\partial \Phi_{vol}}{\partial C}}_{S_{vol}} + \underbrace{2\frac{\partial \Phi_{ich}}{\partial C}}_{S_{ich}}$$

$$S_{vol} = -Jp\,C^{-1} \qquad S_{ich} = J^{-2/3}\,\mathbb{P}:\bar{S} \qquad \bar{S} = 2\frac{\partial \Phi_{ich}}{\partial \bar{C}} \quad \text{fictitious 2nd PK stress}$$

$$p = -\frac{\partial \Phi_{vol}}{\partial J}$$

$$\sigma = J^{-1} F S F^T = \sigma_{vol} + \sigma_{ich}$$

$$\sigma_{vol} = -pI \qquad \sigma_{ich} = J^{-1}\bar{F}(\mathbb{P}:\bar{S})\bar{F}^T$$

↳ traceless. Verify!

↳ σ_{ich} is indeed the deviatoric part of σ.

$$\mathbb{C} := 2\frac{\partial S}{\partial C} = \mathbb{C}_{vol} + \mathbb{C}_{ich}$$

$$\mathbb{C}_{vol} = J\left(p + J\frac{dp}{dJ}\right) C^{-1}\otimes C^{-1} - 2Jp\, C^{-1}\odot C^{-1}$$

$$\left(C^{-1}\odot C^{-1}\right)_{IJKL} = \frac{1}{2}\left(C^{-1}_{IK}C^{-1}_{JL} + C^{-1}_{IL}C^{-1}_{JK}\right) := -\frac{\partial C^{-1}}{\partial C}$$

$$\mathbb{C}_{ich} = \mathbb{P} : \bar{\mathbb{C}} : \mathbb{P}^T + \frac{2}{3} Tr\left(J^{-2/3} \bar{S}\right) \tilde{\mathbb{P}} - \frac{2}{3}\left(C^{-1} \otimes S_{ich} + S_{ich} \otimes C^{-1}\right)$$

$$\bar{\mathbb{C}} := 4 J^{-4/3} \frac{\partial \bar{\Phi}_{ich}}{\partial \bar{C}\, \partial \bar{C}} \qquad Tr(\cdot) := (\cdot) : C$$

$$\tilde{\mathbb{P}} = C^{-1} \odot C^{-1} - \frac{1}{3} C^{-1} \otimes C^{-1}.$$

Remark: Refer to Holzapfel book example 6.8 for the derivation of $\mathbb{C}_{ich}$. Note, the author uses iso for isochoric quantities.

Remark: The formula of $\mathbb{C}$ for <u>stretch based</u> models can be found on Holzapfel book, pp. 257 - 260.

Example: neo-Hookean $\psi_{ich}(\bar{C}) = \overset{\mu/2}{\overset{\shortparallel}{c_1}}(\bar{I}_1 - 3)$

C.H. Liu, G. Hofstetter, H. Mang 1994 $\longrightarrow$ $\psi_{vol}(J) = \kappa(J \ln J - J + 1)$

$$\bar{S} = c_1 I \quad \Rightarrow \quad S_{ich} = J^{-2/3}\left(\mathbb{I} - \frac{1}{3} C^{-1} \otimes C\right) : c_1 I$$

$$= c_1 J^{-2/3}\left(I - \frac{1}{3} \operatorname{tr} C\, C^{-1}\right)$$

$$p = -\frac{\partial \psi_{vol}}{\partial J} = -\kappa \ln J$$

Ogden model. $\psi_{ich}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) = \sum_{a=1}^{3} \bar{\omega}(\bar{\lambda}_a) \qquad \bar{\omega}(\bar{\lambda}_a) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left(\bar{\lambda}_a^{\alpha_p} - 1\right)$

bulk modulus $\longrightarrow$ $\psi_{vol}(J) = \kappa \beta^{-2}\left(\beta \ln J + J^{-\beta} - 1\right)$

$\underline{\underline{\beta = 9}}$