Stress tensors :

$$t_i = \sigma_{ij} n_j \quad \text{or} \quad \vec{t} = \bar{\bar{\sigma}} \vec{n}$$

→ Cauchy stress (true stress): a linear operator that transforms unit normal vector of an area element to the traction acting on the surface.

- $\sigma_{ij} = \sigma_{ji}$ due to the balance of angular momentum (without couple-stresses).
- $\bar{\bar{\tau}} = J \bar{\bar{\sigma}}$ Kirchhoff stress tensor

Now, we introduce the traction on the corresponding initial configuration T_i satisfying $T_i \, dA = t_i \, da$

→ T_i and t_i are parallel. different due to the scaling of areas.

Introduce a two-point tensor P_{iI} such that

$$T_i = P_{iI} N_I$$

→ The first Piola-Kirchhoff stress (PK)

$$P_{iI}\, \underline{\underline{N_I\, dA}} = \sigma_{ij} n_j\, da = \sigma_{ij}\, J\, F^{-1}_{Ij}\, \underline{\underline{N_I\, dA}}$$

$$\boxed{\begin{array}{l} P_{iI} = J\,\sigma_{ij}\, F^{-1}_{Ij} = J\,\sigma_{ij}\,(F^{-T})_{jI} \\ \text{or} \qquad \bar{\bar{P}} = J\,\bar{\bar{\sigma}}\,\bar{\bar{F}}^{-T} = \bar{\bar{\tau}}\,\bar{\bar{F}}^{-T} \end{array}}$$

Let $V = B_\varepsilon$: a spherical ball with radius $\varepsilon > 0$

$v = \varphi_t(B_\varepsilon) = b_\varepsilon$

$s = \partial b_\varepsilon$

$S = \partial B_\varepsilon$

$$\int_{s=\partial b_\varepsilon} \vec{t}\, da = \int_{S=\partial B_\varepsilon} \vec{T}\, dA$$

$$\int_s \sigma n\, da = \int_S P N\, dA$$

$$\int_v \sigma_{ij,j}\, dv = \int_V P_{iI,I}\, dV$$

$$= \int_V \sigma_{ij,j}\, J\, dV$$

Divergence theorem:

$$\int_v \frac{\partial}{\partial x_i}(\dots)\, dv = \int_{\partial v} (\dots)\, n_i\, da$$

$$\int_V \frac{\partial}{\partial X_I}(\dots)\, dV = \int_{\partial V} (\dots)\, N_I\, dA$$

$\Longrightarrow$ localization by shrinking the ball :

$$\boxed{\begin{array}{c} \sigma_{ij,j}\, J = P_{iI,I} \\ \text{or} \\ J\, \operatorname{div}\sigma = \operatorname{DIV} P \end{array}}$$

S is a symmetric material tensor defined as

$$S = F^{-1} P = J F^{-1} \sigma F^{-T} = F^{-1} \tau F^{-T}$$

or $$S_{IJ} = F^{-1}_{Ii} P_{iJ} = J F^{-1}_{Ii} \sigma_{ij} F^{-1}_{Jj} = F^{-1}_{Ii} \tau_{ij} F^{-1}_{Jj}$$

Constitutive Relations:

There exists a strain energy density function per unit undeformed volume:

$$\Phi = \Phi(E) \qquad E = \tfrac{1}{2}(C - I) = \tfrac{1}{2}(F^T F - I)$$

$$= \hat{\Phi}(C) = \breve{\Phi}(F)$$

Constitution: $$S = \frac{\partial \Phi}{\partial E} \quad \text{or} \quad S_{IJ} = \frac{\partial \Phi}{\partial E_{IJ}}.$$

$$\mathbb{C} = \frac{\partial S}{\partial E} = \frac{\partial^2 \Phi}{\partial E \, \partial E} \qquad \mathbb{C}_{IJKL} = \frac{\partial S_{IJ}}{\partial E_{KL}} = \frac{\partial^2 \Phi(E)}{\partial E_{IJ} \, \partial E_{KL}}.$$

rank-four tensor,
elasticity tensor

spatial counterpart:

$$\mathbb{c}_{ijkl} = J^{-1} F_{iI} F_{jJ} F_{kK} F_{lL} \mathbb{C}_{IJKL}$$

push-forwark of rank-4 tensor

Ex. St. Venant - Kirchhoff material.

$$\Phi(E) = \frac{1}{2}\lambda\,(\mathrm{tr}E)^2 + \mu\, E:E$$

$$\frac{\partial\, \mathrm{tr}E}{\partial E} = I. \qquad \frac{\partial\, E:E}{\partial E} = 2E$$

$$S = \frac{\partial \Phi}{\partial E} = \lambda\, \mathrm{tr}E\, I + 2\mu E$$

$$\mathbb{C} = \frac{\partial S}{\partial E} = \lambda\, I \otimes I + 2\mu\, \mathbb{I}$$

or

$$\mathbb{C}_{IJKL} = \lambda\, \delta_{IJ}\delta_{KL} + \mu\,(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK})$$

Remark: 1: oftentimes, we write strain energy function in terms of C. and $S = 2\dfrac{\partial \Phi(C)}{\partial C}$

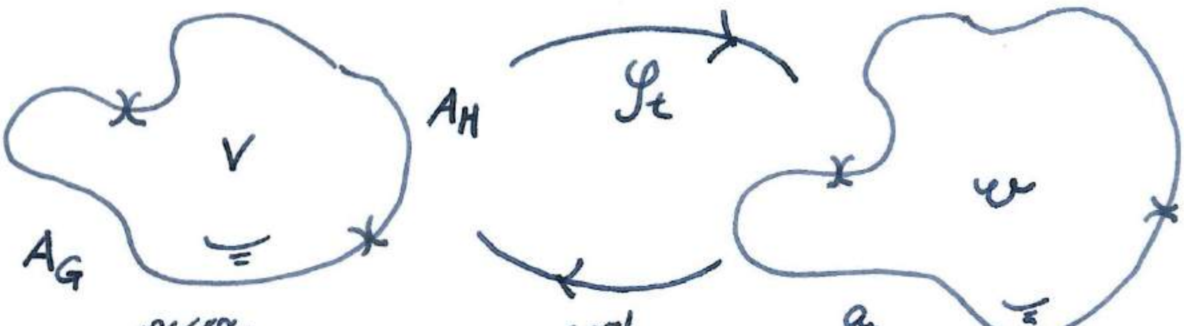
$$\mathbb{C} = 4\frac{\partial^2 \Phi}{\partial C\, \partial C}.$$

2: in the material model, the input is typically F.

$$\begin{tikzcd}
F \arrow[r] & C \arrow[r] \arrow[d, phantom, "\shortparallel"] & E \arrow[r] \arrow[d, phantom, "\shortparallel"] & S \arrow[r] \arrow[dd] & P \arrow[r] \arrow[d, phantom, "\shortparallel"] & \sigma \arrow[d, phantom, "\shortparallel"] \\
& F^T F & \frac{1}{2}(C-I) & & FS & \frac{1}{J} P F^T \\
& & & \mathbb{C} \arrow[r, "" , "\text{push-forward}"'] & c &
\end{tikzcd}$$

- Boundary-value problem

$$\varphi_t(A_H) = a_h$$



$$a_g = \varphi_t(A_G)$$

$$v = \varphi_t(V)$$

$U = G$ on A_G ~~$g \circ \varphi_t$~~

$T = PN = H$ on A_H ~~$h \circ \varphi_t$~~

$u = g$ on $a_g = \varphi_t(A_G)$

$t = \sigma n = h$ on $a_h = \varphi_t(A_H)$

on the current configuration:

$$\operatorname{div} \sigma + f = 0 \quad \text{in } v$$

or

$$\sigma_{ij,j} + f_i = 0$$

body force per unit ~~un~~deformed volume

$$f_i = \rho\, b_i$$

acceleration.

mass per deformed volume

$$\int_{b_\varepsilon = \varphi_t(B_\varepsilon)} \rho(x,t)\, \underbrace{dv}_{J\,dV} = \text{const}$$

$$\|$$

$$\int_{B_\varepsilon} \rho_0(X)\, dV$$

$$\rho(x,t)\, J(X,t) = \rho_0(X) \quad \text{or} \quad \rho_t \circ \varphi_t\, J = \rho_0$$

or $\quad \rho J = \rho_0 \quad$ Local Lagrangian form of mass conservation.

$$J\,(\operatorname{div}\sigma + \rho b) = 0$$

$$\shortparallel$$

$$\mathrm{DIV}\,P + \rho_0 B \longleftarrow B(X,t) = b(\varphi_t(X),t)$$

Weak-form problem:

$$\int_v w_{i,j}\,\sigma_{ij}\,dv = \int_v w_i\,\rho b_i\,dv + \int_{a_h} w_i\,h_i\,da \qquad \forall w_i \in \mathcal{V}_i$$

$$W(X) = w(\varphi_t(X)) \longrightarrow w_{i,j} F_{jI} = W_{i,I}$$

$$\int_V W_{i,I}\,F^{-1}_{Ij}\,\sigma_{ij}\,J\,dV = \int_V W_i\,\rho b_i\,J\,dV + \int_{A_H} W_i\,\overbrace{H_i\,dA}^{h_i\,da\ \shortparallel}$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$\int_V W_{i,I}\,P_{iI}\,dV \qquad\qquad \int_V W_i\,\rho_0 B_i\,dV$$

Remark: $h_i\,da = H_i\,dA$ meaning h_i is parallel to H_i and scaled by a factor $\dfrac{da}{dA} = \sqrt{J^2 F^{-1}_{Ii} N_I\, F^{-1}_{Ji} N_J}$

$$= J\sqrt{N \cdot C^{-1} N}.$$

Remark: if B, G, H do not vary with the deformation, we call them "dead loads".

If we define $\mathcal{F}(\varphi) := \int_V W_{i,I}\, P_{iI}\, dV - \int_V W_i\, \rho_0\, B_i\, dV - \int_{A_H} W_i H_i\, dA$

then the Newton-Raphson method will be

$$D\mathcal{F}(\varphi) \cdot \Delta U = -\mathcal{F}(\varphi)$$

linear action

Can be calculated by the Gateaux derivative:

$$\lim_{\varepsilon \to 0}\left\{\frac{d}{d\varepsilon}\mathcal{F}(\varphi + \varepsilon \Delta U)\right\}$$

$$\mathcal{F}(\varphi + \varepsilon\Delta U) = \int_V W_{i,I}\, \frac{\partial}{\partial X_J}(\varphi_i + \varepsilon \Delta U_i)\, S_{JI}(E(\varphi + \varepsilon\Delta U))\, dV$$

$$\frac{1}{2}\left\{\frac{\partial}{\partial X}(\varphi + \varepsilon\Delta U)^T \frac{\partial}{\partial X}(\varphi + \varepsilon \Delta U) - I\right\}$$

$$\lim_{\varepsilon\to 0}\mathcal{F}(\varphi + \varepsilon\Delta U) = \int_V \underbrace{W_{i,I}\, \Delta U_{i,J}\, S_{JI}}_{\textcircled{1}} + \underbrace{W_{i,I}\, F_{iJ}\, \frac{d}{d\varepsilon}\Big|_{\varepsilon\to 0} S_{JI}(E(\varphi + \varepsilon\Delta U))}_{\textcircled{2}}\, dV$$

$$\frac{d}{d\varepsilon} S_{JI}(E(\varphi + \varepsilon\Delta U)) = \frac{\partial S_{JI}}{\partial E_{KL}}\left\{\frac{1}{2}\Delta U_{k,K} F_{kL} + \frac{1}{2}\Delta U_{k,L} F_{kK}\right\}$$

$$\textcircled{1}: \int_V W_{i,I}\, \Delta U_{i,J}\, S_{JI}\, dV = \int_v w_{i,j} F_{jI}\, \Delta u_{i,l}\, F_{lJ}\, S_{JI}\, J^{-1}\, dv$$

$$= \int_v w_{i,j}\, \sigma_{jl}\, \Delta u_{i,l}\, dv = \int_v w_{i,j}\, \overbrace{\sigma_{jl}\, \delta_{ik}}^{d_{ijkl}}\, u_{k,l}\, dv$$

$d_{ijkl} = d_{klij} = \sigma_{lj}\delta_{ki}$ ← has major symmetry

↳ minor symmetry is lost: $d_{1212} \neq d_{2121}$.

→ it can see the skew part in $w_{i,j}$ and behaves as resistant to rotation

→ Stiffness due to the current stress state. known as the <u>geometrical stiffness.</u>

$$② : \int_V W_{i,I} F_{iJ}\, \mathbb{C}_{JIKL}\, \frac{1}{2}\left(\Delta U_{k,K} F_{kL} + \Delta U_{k,L} F_{kK}\right) dV$$

$$= \int_{\underset{\varphi_t(V)}{\overset{\shortparallel}{v}}} w_{i,j} F_{jI} F_{iJ}\, \mathbb{C}_{JIKL} \underbrace{\Delta U_{k,K} F_{kL}}_{\substack{\shortparallel \\ \Delta u_{k,l} F_{lK}}} J^{-1}\, dv$$

$$= \int_v w_{i,j} \underbrace{\left(J^{-1} F_{iI} F_{jJ} F_{kK} F_{lL}\, \mathbb{C}_{IJKL}\right)}_{c_{ijkl}} \Delta u_{k,l}\, dv$$

major symm: $c_{klij} = J^{-1} F_{kI} F_{lJ} F_{iK} F_{jL}\, \mathbb{C}_{IJKL}$

$= J^{-1} F_{kI} F_{lJ} F_{iK} F_{jL}\, \mathbb{C}_{KLIJ}$

$= c_{ijkl}$.

minor symm: $c_{jikl} = J^{-1} F_{jI} F_{iJ} F_{kK} F_{lL} \mathbb{C}_{IJKL} = c_{ijkl}$

$c_{ijkl} = c_{ijlk}$

$$\int_v w_{i,j}\, c_{ijkl}\, \Delta u_{k,l}\, dv = \int_v w_{(i,j)}\, c_{ijkl}\, \Delta u_{(k,l)}\, dv.$$

① + ②: $\int_v w_{i,j}\, a_{ijkl}\, \Delta u_{k,l}\, dv$

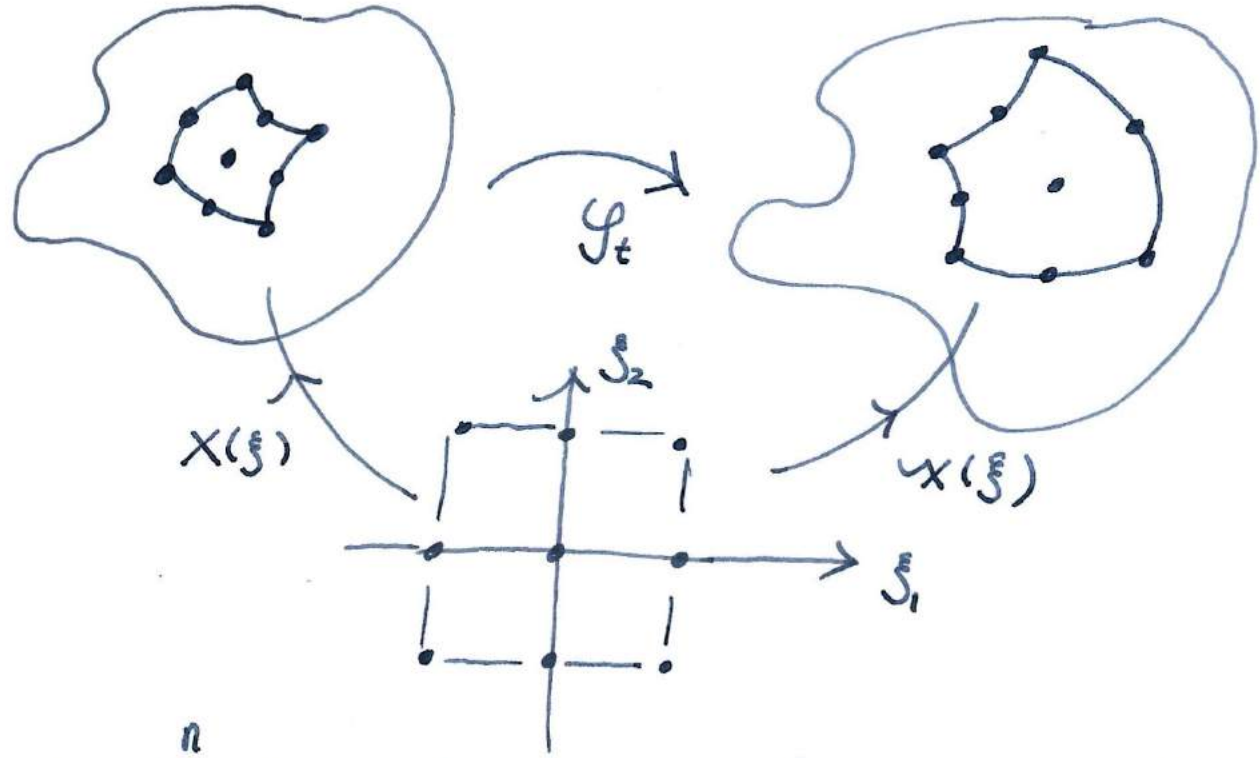
$\to \bar{\bar{a}} = \bar{\bar{c}} + \bar{\bar{d}}$ has major but no minor symmetry

pull-back

$\delta_{ij} S_{IJ} + F_{ik} F_{jL} \mathbb{C}_{IKJL} = J\, F^{-1}_{Ik} F^{-1}_{Jl}\, a_{ikjl}$

$$= \int_V W_{i,I}\, A_{iIjJ}\, \Delta u_{j,J}\, dV$$

One can show $A_{iIjJ} = \dfrac{\partial^2 \phi}{\partial F_{iI}\, \partial F_{jJ}}$.



$$X_I = \sum_{a=1}^{n} N_a(\xi) X^e_{ia}$$

$$x_i = \;..\;\;..\;\; x^e_{ia}$$

$$\varphi_i = \;..\;\;..\;\; (x^e_{ia} \;\cancel{\;\;\;})$$

$$U_i = \;..\;\;..\;\; U^e_{ia}$$

$$u_i = \;..\;\;..\;\; u^e_{ia}$$

$$\int_{V^e} \cdots J\, dV = \int_{v^e} \cdots dv$$

$$\| \qquad\qquad\qquad \|$$

$$\int_{\square} \cdots \underbrace{J}_{\det F} \det\left(\frac{\partial X}{\partial \xi}\right) d\xi_1 d\xi_2 \qquad \int_{\square} \cdots \det\left(\frac{\partial x}{\partial \xi}\right) d\xi_1 d\xi_2$$

$$F^e_{iI} = \frac{\partial \varphi_i}{\partial X_I} = \frac{\partial x^e_i}{\partial X_I} = \frac{\partial x^e_i}{\partial \xi_\alpha} \frac{\partial \xi_\alpha}{\partial X_I}$$

$$= \frac{\partial x^e_i}{\partial \xi_\alpha} \left[\left(\frac{\partial X}{\partial \xi}\right)^{-1}\right]_{\alpha I}.$$

$$D\mathcal{F}(\varphi)\cdot\Delta u = -\mathcal{F}(\varphi) = \overset{③}{\int_{v=\varphi(v)} w_i \rho b_i \, dv} + \overset{④}{\int_{a_h} w_i h_i \, da}$$

$$\parallel$$

$$\overset{①}{\int_{v = \varphi(V)} w_{(i,j)} \mathbb{C}_{ijkl} \Delta u_{(k,l)} \, dv} \quad - \overset{⑤}{\int_v w_{(i,j)} \sigma_{ij} \, dv}$$

$$+$$

$$\overset{②}{\int_v w_{i,j} \delta_{ik} \sigma_{jl} \Delta u_{k,l} \, dv}$$

① : $K^e_{pq} = K^e_{iajb} = e_i^T \int_v B_a^T \mathbb{D} \, B_b \, dv \, e_j$

$$= e_i^T \int_{\square} B_a^T \mathbb{D} \, B_b \det\left(\frac{\partial x}{\partial \xi}\right) \underbrace{d\xi_1 d\xi_2 d\xi_3}_{d\square} e_j$$

② : $K^e_{pq} = K^e_{iajb} = \delta_{ij} K^e_{ab} \quad K^e_{ab} = \int_v N_{a,i} \sigma_{ij} N_{b,j} \, dv.$

③ : $\int_{\square} \underset{J^{-1}\rho_0}{\underbrace{\rho}} N_a^{(\xi)} \underset{b_i(x(\xi))}{\underbrace{b_i}} \det\left(\frac{\partial x}{\partial \xi}\right) d\square$

④ : $\int_{a_h} N_a h_i \, da = \int_{\partial\square} N_a h_i(x(\xi)) \frac{|da|}{|d\partial\square|} \, d\partial\square$

⑤ : $e_i^T \int_{\square} B_a^T \sigma^{vect} \det\left(\frac{\partial x}{\partial \xi}\right) d\square.$

Voigt notation.