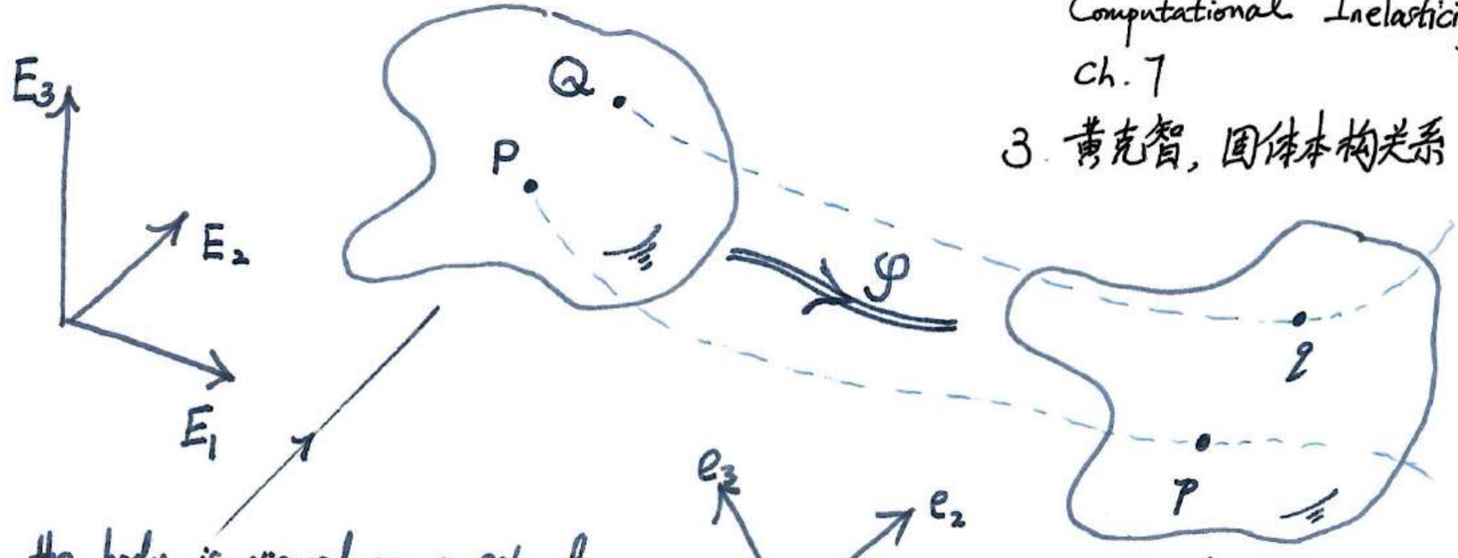
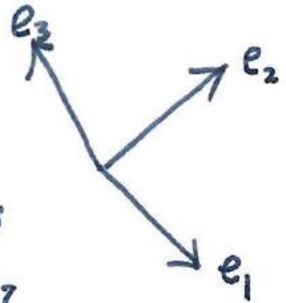
Finite-strain elasticity

References:

1. G. Holzapfel: Nonlinear Solid Mechanics
2. J. Simo & T.J.R. Hughes: Computational Inelasticity ch. 7
3. 黄克智, 固体本构关系

- Kinematics: the study of motion and deformation without reference to the cause.





the body is viewed as a set of particles, labelled by the coordinates (X_1, X_2, X_3) with respect to $\{\vec{E}_I\}$ at time $t=0$.

material points

I, J, K, L

$(\ \ )_{,I} = \dfrac{\partial}{\partial X_I}$.

the current position of the particles at time t is (x_1, x_2, x_3) w.r.t. $\{\vec{e}_i\}$

spatial points i, j, k, l.

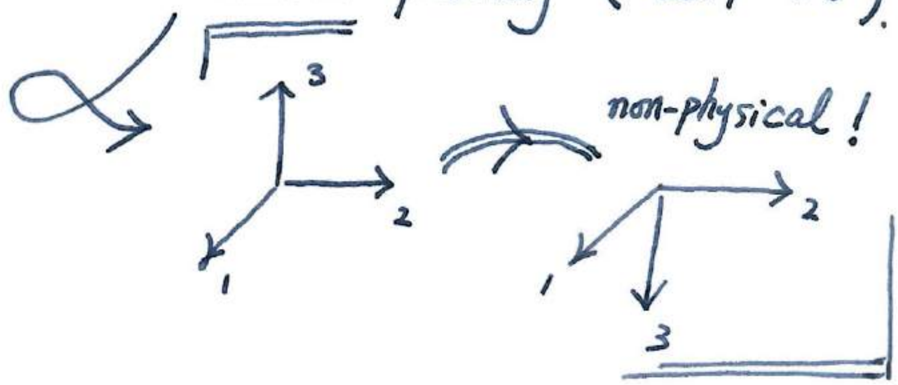
$(\ \ )_{,i} = \dfrac{\partial}{\partial x_i}$

$$\vec{x} = \varphi(\vec{X}, t)$$

placement: gives the configuration Ω_{t} at time t

Requirement: • smooth (differentiable);

- one-to-one (except possibly at the boundary: contact)
- orientation-preserving ($\det F > 0$).



Displacement: $U_i(X, t) = \varphi_i(X, t) - \varphi_i(X, 0)$

$$= x_i - \delta_{iI} X_I$$

a vector on Ω_x

Note: the ambient space is Cartesian

rigid motion: $x = Q(t)\, X + c(t)$

rigid rotation: Q is proper orthogonal.

rigid translation

$\det(Q) = +1 \qquad Q^T Q = I.$

Deformation gradient:

$$dx_1 = x_{q_1} - x_p = \varphi_t(X_{Q_1}) - \varphi_t(X_P)$$

$$= \varphi_t(X_P + dX_1) - \varphi_t(X_P) = \frac{\partial \varphi_t}{\partial X}(X_P)\, dX_1 .$$

We call $\dfrac{\partial \varphi_t}{\partial X} = \dfrac{\partial x}{\partial X} = F$ the <u>deformation gradient</u>.

$$F = F_{iI}\ \vec{e}_i \otimes \vec{E}_I$$

current/spatial Cartesian basis ($\vec{e}_i$)   initial material Cartesian basis ($\vec{E}_I$)

$$F^T = F_{iI}\ \vec{E}_I \otimes \vec{e}_i = (F^T)_{Ii}\ \vec{E}_I \otimes \vec{e}_i$$

$$F^{-1} = \frac{\partial X}{\partial x} = \frac{\partial X_I}{\partial x_i}\ \vec{E}_I \otimes \vec{e}_i$$

$$F^{-T} = \frac{\partial X_I}{\partial x_i}\ \vec{e}_i \otimes \vec{E}_I$$

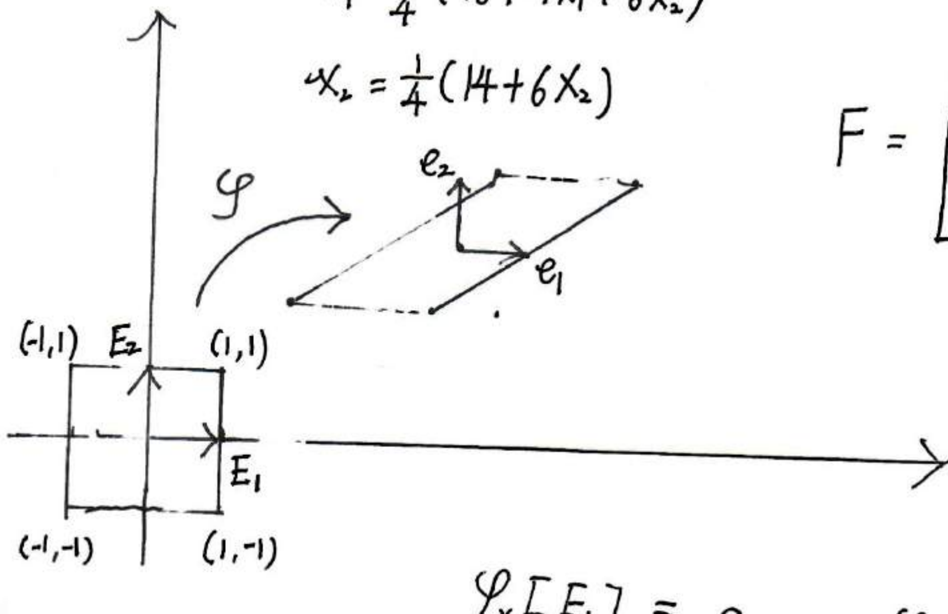
Two-point tensors: transform vectors of one configuration to vectors on another configuration

→ more general:

- push-forward $dx = F\,dX = \varphi_{t*}[dX]$
- pull back $dX = F^{-1} dx = \varphi_{t*}^{-1}[dx]$.

$$x_1 = \frac{1}{4}(18 + 4X_1 + 6X_2)$$

$$x_2 = \frac{1}{4}(14 + 6X_2)$$



$$F = \begin{bmatrix} 1 & 1.5 \\ 0 & 1.5 \end{bmatrix}$$

$$F^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & \frac{2}{3} \end{bmatrix}$$

$$\varphi_*[E_1] = e_1 \qquad \varphi_*[E_2] = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$$

$$\varphi_*^{-1}[e_1] = E_1 \qquad \varphi_*^{-1}[e_2] = \begin{bmatrix} -1 \\ 2 \\ \frac{2}{3} \end{bmatrix}$$

- Line, Area, and Volume change

$$\underline{dx = F\, dX} \quad \text{line element}$$

> **Lemma:** $v, w \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 3}$, then
> $$(Av) \times (Aw) = (\operatorname{cof} A)(v \times w)$$
> if A^{-1} exists, $\operatorname{cof} A = \det A\, A^{-T}$.

Proof:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} M_{ij}$$

minor 余子式

Laplace expansion

$$(-1)^{i+j} M_{ij} = (\operatorname{cof} A)_{ij}$$

cofactor 代数余子式.

$$\Rightarrow \quad (\operatorname{cof} A)_{ij} A^T_{jk} = (\det A)\, \delta_{ik}$$

or if A is invertible, $\operatorname{cof} A = (\det A)\, A^{-T}$.

Levi-Civita symbol:

$$\det A = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} A_{ip} A_{jq} A_{kr}$$

$$v \times w = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\, e_i v_j w_k.$$

$$(\mathrm{cof}A)_{ij}\, \underline{\underline{A_{ij}}} = 3 \det A = \frac{1}{2} \varepsilon_{ipk}\, \varepsilon_{jqr}\, \underline{\underline{A_{ij}}}\, A_{pq} A_{kr}$$

$$\Rightarrow \quad (\mathrm{cof}A)_{ij} = \frac{1}{2} \varepsilon_{ipk}\, \varepsilon_{jqr}\, A_{pq} A_{kr}.$$

$$\Rightarrow (\mathrm{cof}A)_{ij} (v \times w)_j = (\mathrm{cof}A)_{ij}\, \varepsilon_{jmn} v_m w_n$$

$$= \frac{1}{2} \varepsilon_{ipk}\, \varepsilon_{jqr}\, \varepsilon_{jmn}\, v_m w_n\, A_{pq} A_{kr}$$

$$= \frac{1}{2} \varepsilon_{ipk} \left(\delta_{qm} \delta_{rn} - \delta_{qn} \delta_{rm} \right) v_m w_n A_{pq} A_{kr}.$$

$$= \frac{1}{2} \varepsilon_{ipk} \cancel{(V_q W_r - V_r W_q)} \left(V_q W_r A_{pq} A_{kr} - V_r W_q A_{pq} A_{kr} \right)$$

$$= \frac{1}{2} \varepsilon_{ipk} V_q W_r A_{pq} A_{kr} - \frac{1}{2} \varepsilon_{ipk} V_r W_q A_{pq} A_{kr}$$

$$= \varepsilon_{ipk} (A_{pq} V_q)(A_{kr} W_r).$$

$\blacksquare$

Now we may consider an area element:

$$dX_1 \times dX_2 = N\, dA.$$

orientation: normal vector.

after deformation: $dx_1 \times dx_2 = n\, da$

$$\begin{aligned} &\| \\ &(F dX_1) \times (F dX_2) \\ &\| \\ &(\mathrm{cof} F)(dX_1 \times dX_2) \\ &\| \\ &(\mathrm{cof} F)\, N\, dA. \end{aligned}$$

Jacobian $\quad J := \det F.$

$$\boxed{n\,da = J F^{-T} N\,dA}$$ Nanson's formula.

F is invertible.

$$\boxed{\int_{\mathcal{S}} \cdots\ n\,da = \int_{S} \cdots\ J F^{-T} N\,dA.}$$

this is how one change the integration variable for integration on orientable surfaces. (e.g. traction integration).

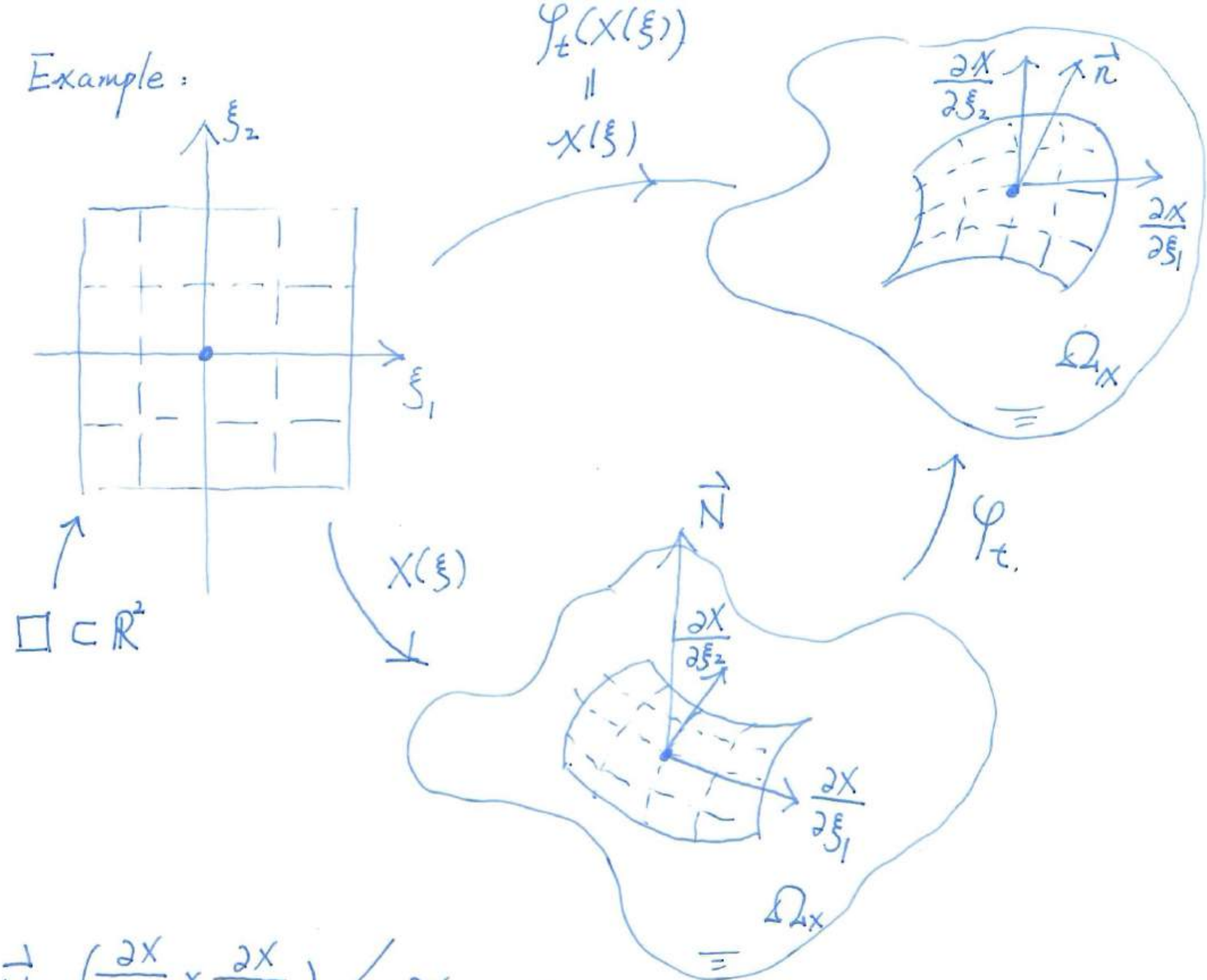
Consider a volume element $\quad dX_1 \cdot (dX_2 \times dX_3)$

after deformation

$$d\mathcal{X}_1 \cdot (d\mathcal{X}_2 \times d\mathcal{X}_3)$$
$$= F dX_1 \cdot (F dX_2 \times F dX_3)$$
$$= dX_1 \cdot (F^T \operatorname{cof} F) (dX_2 \times dX_3)$$
$$= J \ dX_1 \cdot (dX_2 \times dX_3)$$

$$\boxed{\int_{v = \varphi_t(V)} \cdots\ dv = \int_{V} \cdots\ J\,dV}$$

Example:



$$\vec{N} = \left(\frac{\partial X}{\partial \xi_1} \times \frac{\partial X}{\partial \xi_2}\right) \Big/ \left\| \frac{\partial X}{\partial \xi_1} \times \frac{\partial X}{\partial \xi_2} \right\|$$

$$\vec{n} = \left(\frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2}\right) \Big/ \left\| \frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2} \right\|$$

$$\int_S \cdots \, dA = \int_\square \cdots \left\| \frac{\partial X}{\partial \xi_1} \times \frac{\partial X}{\partial \xi_2} \right\| d\xi_1 \, d\xi_2$$

$$\int_{\underset{\varphi_t(S)}{\overset{\parallel}{s}}} \cdots \, da = \int_\square \cdots \left\| \frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2} \right\| d\xi_1 \, d\xi_2$$

More, $\boxed{\dfrac{\partial J}{\partial F_{iI}} = \operatorname{cof} F_{iI} = J F^{-1}_{Ii}}$

Very useful!

Strain.

measures length & angle.

$$dx_1 \cdot dx_2 = F\,dX_1 \cdot F\,dX_2 = dX_1 \cdot \underbrace{(F^T F)}_{C} dX_2$$

$$C_{IJ} = (F^T)_{Ii} F_{iJ} = F_{iI} F_{iJ}$$

material tensor known as the <u>right Cauchy-Green deformation tensor</u>.

Alternatively, $dX_1 \cdot dX_2 = F^{-1} dx_1 \cdot F^{-1} dx_2$

$$= dx_1 \cdot \underbrace{(F^{-T} F^{-1})}_{b^{-1}} dx_2$$

$$b_{ij} = \cancel{(F^T)}\; F_{iI} F_{jI} \quad (b = FF^T)$$

Spatial tensor known as the <u>left Cauchy-Green deformation tensor</u>, or <u>Finger tensor</u>.

$$\frac{1}{2}(dx_1 \cdot dx_2 - dX_1 \cdot dX_2) = dX_1 \cdot E\,dX_2$$

$$E = \frac{1}{2}(C - I) = \frac{1}{2}(F^T F - I)$$

material tensor, <u>Green-Lagrange Strain tensor</u>

$$\frac{1}{2}(dx_1 \cdot dx_2 - dX_1 \cdot dX_2) = dx_1 \cdot e\, dx_2$$

$$e = \frac{1}{2}(I - b^{-1})$$ spatial tensor, Euler-Almansi strain tensor

Remark 1:

$$2E = F^TF - I = \left[\left(\frac{\partial U}{\partial X}\right) + I\right]^T \left[\left(\frac{\partial U}{\partial X}\right) + I\right] - I$$

$$= \underbrace{\left(\frac{\partial U}{\partial X}\right)^T \left(\frac{\partial U}{\partial X}\right)}_{\text{quadratic/nonlinear}} + \underbrace{\left(\frac{\partial U}{\partial X} + \left(\frac{\partial U}{\partial X}\right)^T\right)}_{\text{Linear.}}$$

$$2\varepsilon = \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^T$$ infinitesimal strain / Small strain.

linear appr. of E.

when deformation is small, we do not differentiate x and X.

For engineering materials, $E \sim 2 \times 10^{11}$ Pa.

$\sigma_Y \sim 2 \times 10^8$ Pa $\Rightarrow \sigma = E \underbrace{\frac{\partial U}{\partial X}}_{< 10^{-3} \ll 1}$

thus, it is acceptable to use ε. as the quadratic term will vanish.

When there is rigid rotations.

$$\frac{\partial U}{\partial X} = Q - I \Rightarrow 2E = Q^TQ - I = 0$$

$$2\varepsilon = Q + Q^T - 2I.$$

Consider 2D rotations $Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\mathcal{E} = (\cos\theta - 1)\, I. \simeq -\frac{\theta^2}{2} I. \quad \text{good for small } \theta.$$

extreme case $\theta = \pi/2$. $\mathcal{E} = -I$ (bad!)

More on the deformation gradient:

Spectrum decomposition:

there exist $\{\vec{N}_a\}$ and $\{\vec{n}_a\}$ $a = 1, 2, 3$, and $\{\lambda_a\}$ ($\lambda_a \in \mathbb{R}$)

$\{\vec{N}_a\}$: material, mutually orthogonal normalized

$\{\vec{n}_a\}$: spatial, mutually orthogonal, normalized

such that

$$\boxed{F = \sum_{a=1}^{3} \lambda_a \vec{n}_a \otimes \vec{N}_a}$$

λ_a: principal stretches

$\vec{n}_a$: principal spatial directions/axes

$\vec{N}_a$: principal referential directions/axes

$$\boxed{F^{-1} = \sum_{a=1}^{3} \lambda_a^{-1} \vec{N}_a \otimes \vec{n}_a}$$

$$F \vec{N}_a = \sum_{b=1}^{3} \lambda_b \vec{n}_b (\vec{N}_b \cdot \vec{N}_a) = \lambda_a \vec{n}_a$$

$$F^{-1} \vec{n}_a = \sum_{b=1}^{3} \lambda_b^{-1} \vec{N}_b\, \vec{n}_b \cdot \vec{n}_a = \lambda_a^{-1} \vec{N}_a$$

$$C = \sum_{a=1}^{3} \lambda_a^2 \vec{N}_a \otimes \vec{N}_a \qquad b = \sum_{a=1}^{3} \lambda_a^2 \vec{n}_a \otimes \vec{n}_a$$

$$\Rightarrow \quad E = \frac{1}{2}\sum_{a=1}^{3} (\lambda_a^2 - 1) \vec{N}_a \otimes \vec{N}_a \qquad e = \frac{1}{2}\sum_{a=1}^{3} (1 - \lambda_a^{-2}) \vec{n}_a \otimes \vec{n}_a$$

the notion of strain can be generalized:

$$E^{(n)} = \frac{1}{n} \sum_{a=1}^{3} (\lambda_a^n - 1) \vec{N}_a \otimes \vec{N}_a \qquad e^{(n)} = \frac{1}{n} \sum_{a=1}^{3} (1 - \lambda_a^{-n}) \vec{n}_a \otimes \vec{n}_a .$$

and the logarithmic strain

$$E^{(0)} = \sum_{a=1}^{3} \ln \lambda_a \vec{N}_a \otimes \vec{N}_a \qquad e^{(0)} = \sum_{a=1}^{3} \ln \lambda_a \vec{n}_a \otimes \vec{n}_a .$$

Interesting features.
- Incompressibility : $\operatorname{tr} E^{(0)} = 0$
- Volumetric and ~~additive~~ isochoric parts are additively splited.

Thus, finit strain theory becomes similar to small strain theory.

In practice, we often form $C = F^T F$ and obtain $\{\vec{N}_a\}$ & $\{\lambda_a\}$ by performing eigen-decomposition. Then $\vec{n}_a = F \vec{N}_a / \lambda_a$.

Algorithm: W.M. Scherzinger & C.R. Dohrmann CMAME 197 (2008) 4007 – 4015.

To be open-sourced. → M^3C lab owns a C++ implementation.

Volumetric - Distortional decomposition

$$F = (J^{1/3} I)\, \bar{F}$$

volume-preserving or distortional part.

$$\underbrace{\phantom{J^{1/3} I}}_{\text{Volumetric part}}$$

obviously $\bar{F} = J^{-1/3} F$.

$$\bar{F} = \sum_{a=1}^{3} \bar{\lambda}_a \vec{n}_a \otimes \vec{N}_a \qquad \bar{\lambda}_a = J^{-1/3} \lambda_a$$

also, modified deformation gradient.

modified principal stretches

Ref: J.C. Simo & R.L. Taylor. CMAME 85 (1991) 273-310

$$\bar{C} = \bar{F}^T \bar{F} = J^{-2/3} C$$

$$\bar{b} = \bar{F} \bar{F}^T = J^{-2/3} b.$$

Transformation of tensors:

push-forward operation $\chi_*(\cdot)$: transform a tensor based on the reference configuration to the current configuration.

pull-back operation $\chi_*^{-1}(\cdot)$ or $\chi^*(\cdot)$: transform a tensor based on the current configuration to the reference configuration.

For covariant tensors (E, C, e, b^{-1})

$$\chi_*(\cdot) = F^{-T}(\cdot)F^{-1} \qquad \chi_*^{-1}(\cdot) = F^{T}(\cdot)F$$

e.g. $$\chi_*(E) = F^{-T}EF^{-1} = F^{-T}\frac{1}{2}(F^TF - I)F^{-1}$$
$$= \frac{1}{2}(I - b^{-1}) = e$$

For contravariant tensors (C^{-1}, b, most stress tensors)

$$\chi_*(\cdot) = F(\cdot)F^{T} \qquad \chi_*^{-1}(\cdot) = F^{-1}(\cdot)F^{-T}.$$

$$\chi_*^{-1}(b) = F^{-1}(b)F^{-T} = F^{-1}FF^{T}F^{-T} = I$$

metric tensor (in Euclidean space): tells how to get length and angle. inner product

For covariant vectors: $\chi_*(\cdot) = F^{-T}(\cdot) \qquad \chi_*^{-1}(\cdot) = F^{T}(\cdot)$

For contravariant vectors: $\chi_*(\cdot) = F(\cdot) \qquad \chi_*^{-1}(\cdot) = F^{-1}(\cdot)$

Piola transformation: $J\,\chi_*^{-1}(\cdot)$.
for a spatial vector.

Velocity: $$V(X,t) = \frac{\partial}{\partial t}\Big|_X \varphi(X,t)$$ ← Lagrangian or material velocity.
$$= \frac{\partial}{\partial t}\Big|_X U(X,t)$$

Sometimes, we designate $(\dot{\ }) \equiv \frac{\partial}{\partial t}\Big|_X (\ )$
$$V = \dot{\varphi} = \dot{U}$$

Acceleration: $$A(X,t) = \frac{\partial}{\partial t}\Big|_X V(X,t) = \frac{\partial^2}{\partial t^2}\Big|_X \varphi(X,t)$$
$$= \frac{\partial^2}{\partial t^2} U(X,t).$$

define Eulerian velocity & acceleration as

$$v(x,t) = V(X,t) \quad \text{with } x = \varphi_t(X) = \varphi(X,t)$$

or $$v(\varphi(X,t),t) = V(X,t)$$

"drop" t : $\underline{v_t \circ \varphi_t = V_t}$

$$\frac{\partial}{\partial x} v(x,t) = \text{grad}_x v(x,t) = \nabla_x v(x,t) \equiv \underset{\uparrow}{l}$$

spatial velocity gradient

Now we may calculate $\dot{F}$:

$$\dot{F} = \frac{\partial}{\partial t}\Big|_X F = \frac{\partial}{\partial t}\Big|_X \frac{\partial}{\partial X}\Big|_t \varphi(X,t) = \frac{\partial}{\partial X}\Big|_t \frac{\partial}{\partial t}\Big|_X \varphi(X,t)$$

$$= \frac{\partial}{\partial X}\Big|_t V(X,t)$$

$\text{Grad}_X V$ material velocity gradient.

$$\dot{F} = \frac{\partial}{\partial X}\Big|_t V(X,t) = \frac{\partial}{\partial X}\Big|_t v(\varphi_t(X), t)$$

$$= \text{grad}_x v(x,t) \frac{\partial \varphi_t}{\partial X} = l F$$

$$\Rightarrow \quad l = \dot{F} F^{-1} \qquad l_{ij} = \dot{F}_{iI} F^{-1}_{Ij}$$

$$\dot{\overline{F^{-1}}} = - F^{-1} l \qquad \dot{\overline{F^{-1}_{Ii}}} = - F^{-1}_{Ij} l_{ji}$$

$$\dot{\overline{F^{-T}}} = - l^T F^{-T} \qquad \dot{\overline{F^{-T}_{iI}}} = - l_{ji} F^{-1}_{Ij}$$

$$l = d + w \longrightarrow w = \frac{1}{2}(l - l^T)$$

spin tensor, or
rate of rotation tensor,
or vorticity tensor.

$$\longrightarrow d = \frac{1}{2}(l + l^T)$$

rate of deformation tensor
or rate of strain tensor
(in fluid mech. often denoted
by ε).

<u>Material time derivative of strain tensors.</u>

$$\dot{E} = \cancel{\tfrac{1}{2}(\dot{F}^T F + F^T \dot{F})} \; \frac{1}{2}\left(\dot{\overline{F^T}} F + F^T \dot{F} \right)$$

$$= \frac{1}{2}\left(F^T l^T F + F^T l F \right)$$

$$= F^T d F.$$

or simply $\dot{E} = \chi_*^{-1}(d)$

pull-back of covariant
tensor d.

Apparently, $\dot{C} = 2F^T d F.$

$$\dot{b} = lb + bl^T$$

$$\dot{e} = d - l^T e - e d.$$

Lie time derivative.

Consider a spatial field $f(x, t)$. (physical scalar, vector, or tensor quantity). Its Lie time derivative is obtained in the following 3 steps:

1. pull f back to the reference configuration

$$\underbrace{\mathcal{F}(X, t)}_{\text{associated material field.}} = \chi_*^{-1}(f(x, t))$$

2. take material time derivative : $\dot{\mathcal{F}}$

3. push $\dot{\mathcal{F}}$ forward to the current configuration.

Thus. $$\mathcal{L}(f) = \chi_*\left(\frac{D}{Dt}\chi_*^{-1}(f)\right) = \chi_*(\dot{\mathcal{F}}).$$

e.g. $$\mathcal{L}(e) = F^{-T}\left(\frac{D}{Dt}(F^T e F)\right)F^{-1}$$

$$= F^{-T}\,\dot{E}\,F^{-1}$$

$$= d.$$

Lie derivative of the Euler-Almansi strain is the rate of deformation.

$$\mathcal{L}(\underline{\underline{b}}) = F\left(\frac{D}{Dt}(F^{-1} b F^{-T})\right)F^T = \underline{\underline{0}}$$