• Small-strain nonlinear elastostatics

Assumptions :

1. the deformation is small $\longrightarrow$ small strain is adopted.

   u_i : displacement

   ε_{ij} : strain $= u_{(i,j)} := \frac{1}{2}(u_{i,j} + u_{j,i})$

   σ_{ij} : Cauchy / True stress ($\sigma_{ij} = \sigma_{ji}$)

2. the body is initially unstressed (initial configuration is the natural configuration)

3. the material is hyper-elastic, meaning there is a strain energy $\Phi(\varepsilon, x)$ such that

$$\sigma_{ij} = \frac{\partial \Phi}{\partial \varepsilon_{ij}} .$$

The <u>material moduli</u> are defined as

$$C_{ijkl} := \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 \Phi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}$$

e.g. $\Phi := \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$ with $C_{ijkl}(x)$ defines linear elastic materials.

• Further, if $C_{ijkl}(x) = \lambda(x)\,\delta_{ij}\delta_{kl} + \mu(x)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$

the body is <u>isotropic</u>

Lamé parameters

Symmetries :

Minor symmetry $C_{ijkl} = C_{jikl} = C_{ijlk}$

Major symmetry $C_{ijkl} = C_{klij}$

f_i : forces per unit volume e.g. $\rho\, g_i$

density gravitational acceleration

g_i : prescribed displacement on Γ_g

h_i : prescribed traction on Γ_h

Stress vector

(S) Given $f : \Omega \to \mathbb{R}^{n_{sd}}$, $g : \Gamma_g \to \mathbb{R}^{n_{sd}}$, $h : \Gamma_h \to \mathbb{R}^{n_{sd}}$ find the displacement $u : \bar{\Omega} \to \mathbb{R}^{n_{sd}}$ and the stress $\sigma : \bar{\Omega} \to \mathbb{R}^{n_{sd}} \times \mathbb{R}^{n_{sd}}$ s.t.

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega$$
$$u_i = g_i \quad \text{on } \Gamma_g$$
$$\sigma_{ij} n_j = h_i \quad \text{on } \Gamma_h$$

$$
(W)\begin{cases}
\text{Given } f,\ g, \text{ and } h, \text{ find } u_i \in \mathcal{S}_i := \{ u_i : u_i = g_i \text{ on } \Gamma_g \ \cdots \} \\
\text{s.t. for } \forall\ w_i \in \mathcal{V}_i := \{ w_i : w_i = 0 \text{ on } \Gamma_g, \cdots \} \\
\int_\Omega w_{i,j}\, \sigma_{ij}\, d\Omega = \int_\Omega w_i f_i \, d\Omega + \int_{\Gamma_h} w_i h_i \, d\Gamma.
\end{cases}
$$

Euler-Lagrange form :

$$
0 = \int_\Omega w_i (\sigma_{ij,j} - f_i)\, d\Omega - \int_{\Gamma_h} w_i (\sigma_{ij} n_j - h_i)\, d\Gamma
$$

$(S) \Leftrightarrow (W)$

equilibrium in interior & on surface are built into the variational problem.

$$
a(w, u) := \int_\Omega w_{i,j}\, \sigma_{ij}\, d\Omega = \int_\Omega w_{(i,j)}\, \sigma_{ij}\, d\Omega
$$

Claim: S is a second-order tensor,

T is a symmetric second-order tensor.

$$
S_{ij} T_{ij} = S_{(ij)} T_{ij} = S_{(ij)} T_{(i,j)}
$$

Euclidean decomposition: $S_{ij} = S_{(ij)} + S_{[ij]}$

$$S_{(ij)} = \frac{1}{2}\left(S_{ij} + S_{ji} \right) \qquad S_{[ij]} = \frac{1}{2}\left(S_{ij} - S_{ji} \right)$$

Remark: $a(\cdot,\cdot)$ is linear w.r.t. its first slot. in general.

When the material is linear elastic, one has

$$a(w, u) = \int_{\Omega} w_{(i,j)} C_{ijkl} u_{(k,l)} \, d\Omega$$

<u>Galerkin formulation</u>. $\mathcal{V}_i^h \subset \mathcal{V}_i \quad \mathcal{S}_i^h \subset \mathcal{S}_i$

$$\mathcal{V}_i^h = \left\{ w_i^h : w_i^h = 0 \text{ on } \Gamma_g, \; w_i^h = \sum_{A \in \eta - \eta_g} N_A(x) \underset{\mathbb{R}}{\underset{\cap}{C_{iA}}} \right\}$$

$$w^h = w_i^h e_i$$

$$a(w^h, u^h) - (w^h, f) - (w^h, h)_{\Gamma_h} = 0$$

$$\Rightarrow \sum_{A \in \eta - \eta_g} \left\{ C_{iA} \left[a(N_A e_i, u^h) - (N_A e_i, f) - (N_A e_i, h)_{\Gamma_h} \right] \right\} = 0$$

for any C_{iA}.

$$\Rightarrow a(N_A e_i, u^h) - \underbrace{(N_A e_i, f)}_{\int_{\Omega} N_A f_i \, d\Omega} - \underbrace{(N_A e_i, h)_{\Gamma_h}}_{\int_{\Gamma_h} N_A h_i \, d\Gamma} = 0$$

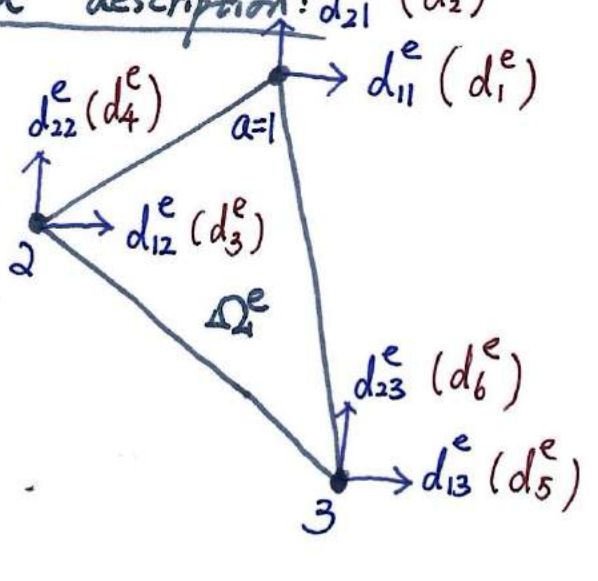
$$ID(i, A) = \begin{cases} P & \text{if node } A \in \eta_{g_i} \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \quad (N) \begin{cases} F^{int} \equiv N(d) = F^{ext} \quad \text{or} \\ N_P(d) = F_P^{ext} \qquad 1 \le P \le n_{eq}. \\ \text{where} \quad N = \underset{e=1}{\overset{n_{el}}{\mathbf{A}}} n^e \quad F^{ext} = \underset{e=1}{\overset{n_{el}}{\mathbf{A}}} f^e. \end{cases}$$

Solving (N) necessitates the introduction of a (M) problem: $\underline{DN(d)\ \Delta d = F^{ext} - N(d)}$

linearization w.r.t. unknown disp. dofs.

Local description:



$$d_{ia}^e \longleftrightarrow d_p^e$$

$$p = 2(a-1) + i$$

n_{sd}

$$1 \le p \le n_{ee}$$

\# of element eqns.

element force vector $\quad f_p^e = \int_{\Omega^e} N_a f_i \, d\Omega + \int_{\Gamma_h^e} N_a h_i \, d\Gamma.$

element internal force $\quad n_p^e = \int_{\Omega^e} (N_a e_i)_{,j} \, \sigma_{ij} \, d\Omega$

$$= \int_{\Omega^e} N_{a,j} \, e_i \, \sigma_{ij} \, d\Omega$$

Voigt notation: vector $\rightarrow$ ...

Second-order tensor $\rightarrow$ "vector" or array

fourth-order tensor $\rightarrow$ "matrix"

collapse a pair of indices to a single index

Strain vector $\varepsilon^{vect}(\omega) := \begin{Bmatrix} \omega_{1,1} \\ \omega_{2,2} \\ \omega_{1,2} + \omega_{2,1} \end{Bmatrix} = \{ \varepsilon^{vect}_I \}$

Stress vector $\sigma^{vect} := \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \{ \sigma^{vect}_I \}$

$$\varepsilon^{vect}(N_a e_i) = \underbrace{\begin{bmatrix} N_{a,1} & \\ & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix}}_{B_a} \underbrace{\begin{Bmatrix} \delta_{1i} \\ \delta_{2i} \end{Bmatrix}}_{e_i}$$

We have $\displaystyle n^e_p = \int_{\Omega_4} \varepsilon^{vect}(N_a e_i)^T \cdot \sigma^{vect} \, d\Omega_4$

$$= e_i^T \int_{\Omega_4} B_a^T \sigma^{vect} \, d\Omega_4$$

$$D n^e_p = \left[\frac{\partial n^e_p}{\partial d^e_q} \right] = \left[\frac{\partial n^e_{ia}}{\partial d^e_{jb}} \right]$$

pq - entry of element tangent matrix.

$$\frac{\partial n_p^e}{\partial d_q^e} = \frac{\partial}{\partial d_{jb}^e}\left(e_i^T \int_{\Omega_1^e} B_a^T \sigma^{vect}(\varepsilon)\, d\Omega_1 \right)$$

$$= e_i^T \int_{\Omega_1^e} B_a^T \frac{\partial}{\partial d_{jb}^e} \sigma^{vect}(\varepsilon)\, d\Omega_1$$

$$= e_i^T \int_{\Omega_1^e} B_a^T \frac{\partial \sigma^{vect}}{\partial \varepsilon^{vect}} \frac{\partial}{\partial d_{jb}^e}\left(B_c d_c^e \right) d\Omega_1$$

$$= e_i^T \int_{\Omega_1^e} B_a^T D\, B_c\, \delta_{cb}\, e_j\, d\Omega_1$$

$$= e_i^T \int_{\Omega_1^e} B_a^T D B_b\, d\Omega\, e_j$$

$$D_{IJ} = C_{ijkl}(\varepsilon) = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}.$$

I/J	i/k	j/l
1	1	1
2	2	2
3	1	2
3	2	1

Remark: In elasticity, DN is formally identical to the linear stiffness (See Hughes book Ch 2). DN is symmetric and positive-definite.

In nonlinear heat eqn. the symmetry is lost, not to mention the positive definiteness.