Element / local point of view

Observation : $N_A = 0$ outside of a neighborhood of node A.

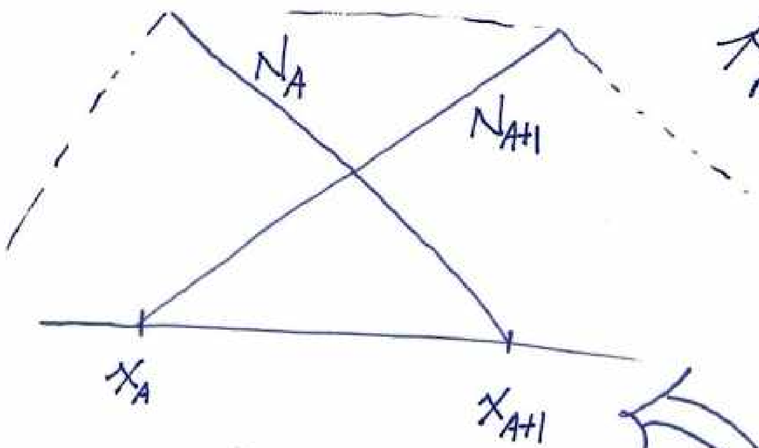
$\Rightarrow$ We only need to consider the neighborhood of node A when performing integration involving N_A.

Simplify the computer implementation.

We shall restrict ourselves to 1D linear element and we will generalize the idea to multi-D and higher-order later.

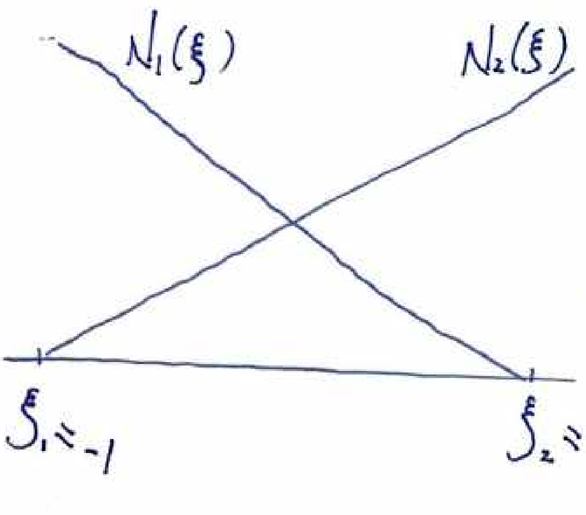
Recall in the 1D element $[x_A, x_{A+1}]$

nodes



$$u^h = N_A d_A + N_{A+1} d_{A+1}$$

for $x \in [x_A, x_{A+1}]$.



$$x = x(\xi) = \frac{\xi h_A + x_A + x_{A+1}}{2}$$

$$\xi = \xi(x) = \frac{2x - x_A - x_{A+1}}{h_A}$$

$$N_1 = \frac{1-\xi}{2}$$

$$N_2 = \frac{1+\xi}{2}$$

or $N_a(\xi) = \dfrac{1+\xi_a \xi}{2}$, $a = 1, 2$

Verify that: 1) $N_A(x) = \underset{\substack{\| \\ 1}}{N_a} \circ \xi(x) \qquad N_{A+1}(x) = N_2 \circ \xi(x)$

2) $x(\xi) = \underset{\substack{\| \\ x_1^e}}{x_A} N_1(\xi) + \underset{\substack{\| \\ x_2^e}}{x_{A+1}} N_2(\xi)$.

$$K = a(N_A, N_B) = \int_0^1 N_{A,x} N_{B,x}\, dx = \sum_{e=1}^{n_{el}} \int_{\Omega^e} N_{A,x} N_{B,x}\, dx$$

$$= \sum_{e=1}^{n_{el}} K^e \quad \text{with} \quad K^e = \left[\int_{\Omega^e} N_{A,x} N_{B,x}\, dx\right].$$

Stiff with size $n_{eq} \times n_{eq}$, K^e: has non-zero entries only for N_A, N_B has support in Ω^e.

For 1D linear element:

in $\Omega^e = [x_c, x_{c+1}]$ we only need to consider $A = c, c+1$
$B = c, c+1$.

$$K^e = [K^e_{ab}]_{2\times2} \qquad K^e_{ab} = \int_{\Omega^e} \underset{\substack{\| \\ 1,2}}{N_{a,x}} \underset{\substack{\| \\ 1,2}}{N_{b,x}}\, dx$$

$$= \int_{-1}^1 N_{a,\xi} \frac{\partial \xi}{\partial x} N_{b,\xi} \frac{\partial \xi}{\partial x} \frac{\partial x}{\partial \xi}\, d\xi$$

1D change of variable formula

$$\boxed{\int_{x_1}^{x_2} f(x)\,dx = \int_{\xi_1}^{\xi_2} f(x(\xi))\, x_{,\xi}(\xi)\, d\xi \quad \text{if } x: [\xi_1, \xi_2] \to [x_1, x_2] \text{ is differentiable}}$$

$$= \int_{-1}^{1} N_{a,\xi} N_{b,\xi} \frac{\partial \xi}{\partial x} \, d\xi$$

$$= \int_{-1}^{1} \frac{\xi_a}{2} \frac{\xi_b}{2} \frac{2}{h_A} \, d\xi$$

$$= \frac{\xi_a \xi_b}{h_A}$$

$$= \frac{(-1)^{a+b}}{h_A}.$$

$$k^e = \begin{bmatrix} 1/h_A & -1/h_A \\ -1/h_A & 1/h_A \end{bmatrix}.$$

$$K = \begin{bmatrix}
\frac{1}{h_1} & \frac{-1}{h_1} & & & & & \\
-\frac{1}{h_1} & \frac{1}{h_1}+\frac{1}{h_2} & -\frac{1}{h_2} & & & & \\
 & \ddots & \ddots & \ddots & & & \\
 & & -\frac{1}{h_{A-1}} & \frac{1}{h_{A-1}}+\frac{1}{h_A} & \frac{-1}{h_A} & & \\
 & & & -\frac{1}{h_A} & \frac{1}{h_A}+\frac{1}{h_{A+1}} & \frac{-1}{h_{A+1}} & \\
 & & & & \ddots & \ddots & \ddots \\
 & & & & & \frac{-1}{h_{n-1}} & \frac{1}{h_{n-1}}
\end{bmatrix}$$

Two data structures:

$$\mathrm{IEN}(\underbrace{a}_{\text{local node number}}, \underbrace{e}_{\text{element number}}) = \overbrace{A}^{\text{global node number}}$$

$$LM(a,e) = ID(IEN(a,e)).$$

Summary: In each element, there are n_{en} nodes / nonzero basis functions. We may denote them as $\{N_a\}_{a=1}^{n_{en}}$, and it is often convenient to pull them back onto a referential element, on which $N_a = N_a(\xi)$ with the mapping $\xi = \xi(x)$.

On each element, we may build a small matrix K^e, known as the element / local stiffness matrix.

$$K^e = [K^e_{ab}] \qquad K^e_{ab} = a(N_a, N_b).$$

and

$$K_{PQ} \leftarrow K_{PQ} + K^e_{ab} \qquad \begin{aligned} P &= LM(a,e) \neq 0 \\ Q &= LM(b,e) \neq 0. \end{aligned}$$

adding the contribution of element e.

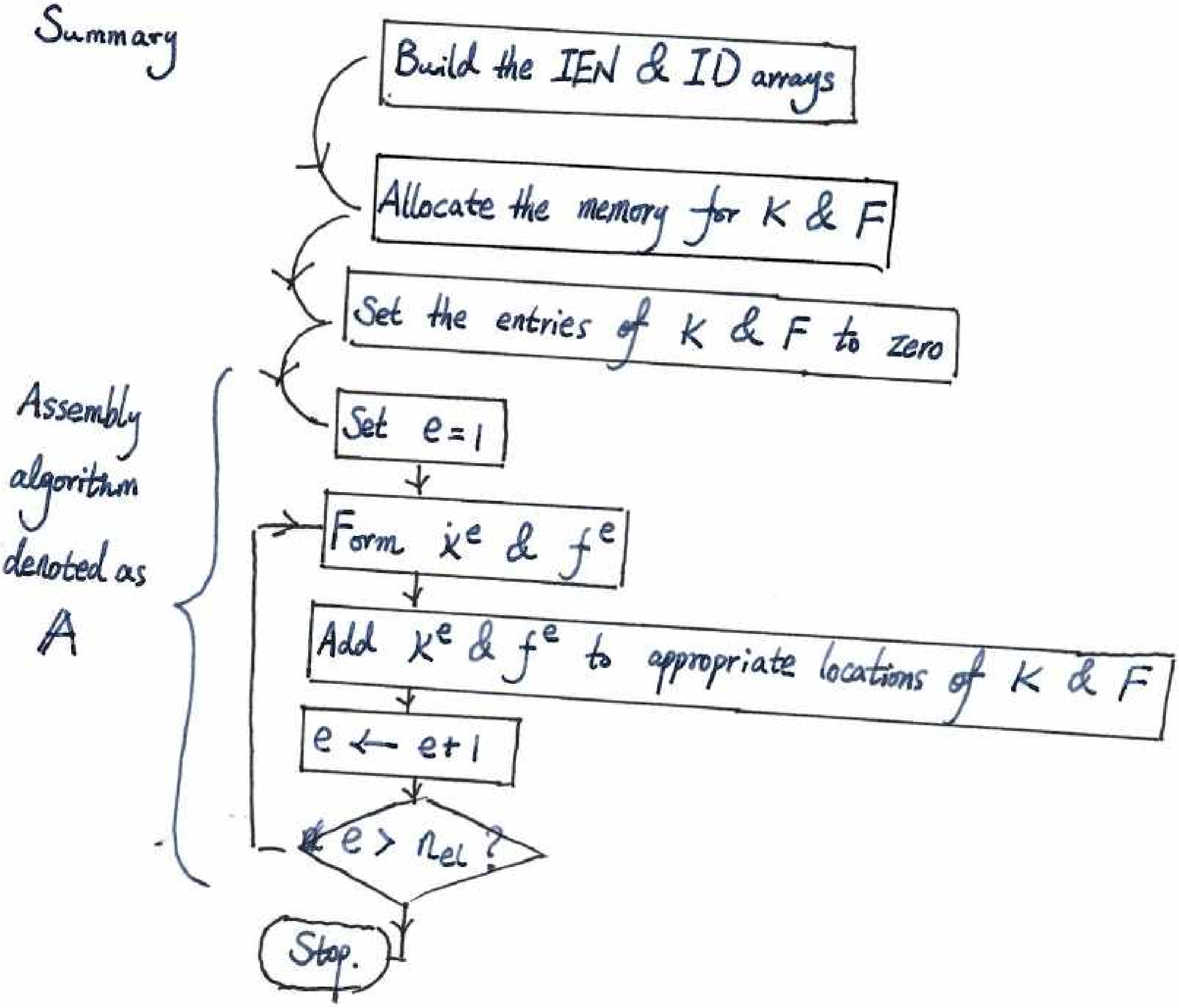
For the load vector

$$F = \sum_{e=1}^{n_{el}} F^e = \sum_{e=1}^{n_{el}} \left\{ (N_A, f)^e + (N_A, h)^e_{\Gamma_h} - a(N_A, N_B)^e g_B \right\}$$

$$f_a^e = \int_{\Omega^e} N_a f \, d\Omega + \int_{\Gamma_h^e} N_a h \, d\Gamma - \sum_{b=1}^{n_{en}} k_{ab}^e g_b^e$$

$g_b^e = g(x_b^e)$ if g is prescribed at x_b^e, otherwise 0.

Summary



$$K = \mathop{\mathsf{A}}_{e=1}^{n_{el}} k^e \quad \& \quad F = \mathop{\mathsf{A}}_{e=1}^{n_{el}} f^e.$$

Constructing the element stiffness matrix & load vector

We gained ~~an~~ first experience in the local / element assembly from the 1D example, in which we applied the chain rule and change-of-variable formula. In multi-dimensional cases, we need to generalize the two and introduce the concept of quadrature.

1. Quadrature.

Let $f : \Omega^e_4 \subset \mathbb{R}^{n_{sd}} \to \mathbb{R}$ be given, we are interested in computing

$$\int_{\Omega^e_4} f(x)\, d\Omega_4 .$$

We always pull the integrand back to the referential / parent element;

$n_{sd} = 1$: $$\int_{\Omega^e_4} f(x)\, dx = \int_{-1}^{1} f(x(\xi))\, x_{,\xi}(\xi)\, d\xi$$

$n_{sd} = 2$: $$\int_{\Omega^e_4} f(x,y)\, d\Omega_4 = \int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta), y(\xi,\eta))\, j(\xi,\eta)\, d\xi d\eta$$

$n_{sd} = 3$ is analogous to the case of $n_{sd} = 2$. Refer to p. 140.

We only need to design an approach for numerical integration over the 'fixed' parent element.

$j = \det\left[\frac{\partial x}{\partial \xi}\right]$. the Jacobian determinant.

For $n_{sd} = 1$ $$\int_{-1}^{1} f(\xi)\,d\xi = \sum_{l=1}^{n_{int}} f(\tilde{\xi}_l) W_l + R$$ ← remainder.

$$\simeq \sum_{l=1}^{n_{int}} f(\tilde{\xi}_l) W_l.$$

number of quadrature points

← l-th weight

coordinate of the l-th quadrature pt

e.g. $n_{int} = 1$, $\tilde{\xi}_1 = 0$, $W_l = 2$, $R = \dfrac{f_{,\xi\xi}(\bar{\xi})}{3}$

$n_{int} = 2$, $\tilde{\xi}_1 = -1/\sqrt{3}$ $W_1 = 1$

$\tilde{\xi}_2 = 1/\sqrt{3}$ $W_2 = 1$ $R = \dfrac{f^{(4)}(\bar{\xi})}{135}$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta \simeq \sum_{l^{(1)}=1}^{n_{int}^{(1)}} \sum_{l^{(2)}=1}^{n_{int}^{(2)}} f(\tilde{\xi}_{l^{(1)}}, \tilde{\eta}_{l^{(2)}}) W_{l^{(1)}}^{(1)} W_{l^{(2)}}^{(2)}$$

$$= \sum_{l=1}^{n_{int}} f(\tilde{\xi}_l, \tilde{\eta}_l) W_l.$$

2. Shape function subroutines

Chain rule : $N_{a,x} = N_{a,\xi}\,\underline{\underline{\xi_{,x}}} + N_{a,\eta}\,\underline{\underline{\eta_{,x}}}$ $\quad (n_{sd}=2)$

$$N_{a,y} = N_{a,\xi}\,\underline{\underline{\xi_{,y}}} + N_{a,\eta}\,\underline{\underline{\eta_{,y}}}$$

Isoparametric element : Let $x : \hat{\Omega} \to \Omega^e$ be of the form

$$x(\xi) = \sum_{a=1}^{n_{en}} N_a(\xi)\, x_a^e$$

and if the local interpolation function is of the form

$$u^h(\xi) = \sum_{a=1}^{n_{en}} N_a(\xi)\, d_a^e$$

the element is <u>isoparametric</u>.

If the element is isoparametric, we have

$$x(\xi,\eta) = \sum_{a=1}^{n_{en}} N_a(\xi,\eta)\, x_a^e$$

$$y(\xi,\eta) = \sum_{a=1}^{n_{en}} N_a(\xi,\eta)\, y_a^e$$

$$x_{,\xi} = \sum_{a=1}^{n_{en}} N_{a,\xi}(\xi,\eta)\, x_a^e$$

$x_{,\eta} = \cdots N_{a,\eta} \cdots \cdots \cdots$

$y_{,\xi} = \cdots N_{a,\xi} \cdots \cdots y_a^e$

$y_{,\eta} = \cdots N_{a,\eta} \cdots \cdots y_a^e$

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}^{-1}$$

Cramer's rule:

$$\begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}^{-1} = \frac{1}{j}\begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix}$$

$$j = \det\left[\vec{x}_{,\vec{\xi}}\right] = x_{,\xi}\, y_{,\eta} - x_{,\eta}\, y_{,\xi}.$$

Now, the step of forming k^e & f^e can be detailed as.

Form k^e & f^e :

For $l = 1, \cdots, n_{int}$

Determine $\tilde{\xi}_l, \tilde{\eta}_l, w_l$

~~For~~ Calculate $x_{,\xi}(\tilde{\xi}_l, \tilde{\eta}_l)$, $x_{,\eta}(\tilde{\xi}_l, \tilde{\eta}_l)$ $y_{,\xi}(\tilde{\xi}_l, \tilde{\eta}_l)$, $y_{,\eta}(\tilde{\xi}_l, \tilde{\eta}_l)$.

Calculate $j(\tilde{\xi}_l, \tilde{\eta}_l)$

Calculate $N_{a,x}(\tilde{\xi}_l, \tilde{\eta}_l)$ $N_{a,y}(\tilde{\xi}_l, \tilde{\eta}_l)$.

Calculate k^e_{ab} & f^e_a

End-for-loop

Reference: • Ch. 3.8 & 3.9 Hughes book.

• github.com/M3C-Lab/FEM-1D-demo.