Galerkin's approximation method.

We construct / design / provide finite-dimensional approximations of $\mathcal{S}$ and $\mathcal{V}$:

$$\mathcal{S}^h \subset \mathcal{S} \qquad \mathcal{V}^h \subset \mathcal{V}.$$

the finite dimensional space is associated with a mesh

Suppose we have a function g^h that satisfies the essential B.C., i.e.,

$$g^h \Big|_{\Gamma_g} = g ,$$

We may construct $\mathcal{S}^h$ from $\mathcal{V}^h$:

$$\mathcal{S}^h = \{ u^h : u^h = v^h + g^h, \ v^h \in \mathcal{V}^h \}$$

$\mathcal{S}^h$ & $\mathcal{V}^h$ are essentially the same set of functions up to the g^h function.

$\Uparrow$

<u>Bubnov</u> - Galerkin Method.

Otherwise, it is called the <u>Petrov</u> - Galerkin Method.

Now, we have an approximated problem:

$$(G)\begin{cases} \text{Given } f:\Omega\to\mathbb{R},\ h:\Gamma_h\to\mathbb{R},\ g:\Gamma_g\to\mathbb{R},\ x:\Omega_x\to\mathbb{R}^{n_{sdx}}_{n_x} \\ \text{find } u^h\in\mathcal{S}^h \text{ s.t. for } \forall\ w^h\in\mathcal{V}^h \\ \qquad a(w^h,u^h)=(w^h,f)+(w^h,h)_{\Gamma_h}. \end{cases}$$

Apparently, (G) and (W) are not equivalent, and we denote their relation by

$$(W)\simeq(G).$$

Notice that (G) is a finite-dimensional linear problem, and we can present it using linear algebra.

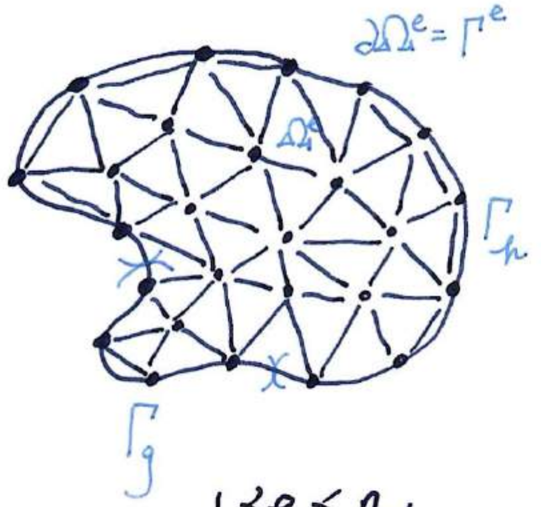
Let ~~functions~~ $N_A(\vec{x})$ ~~by~~, $A=1,2,\cdots,n_{eq}$ be functions in $\mathcal{V}^h$,

dim. of $\mathcal{V}^h$

and any ~~fun~~ $v^h\in\mathcal{V}^h$ can be represented as $v^h=\sum_{A=1}^{n_{eq}}c_A N_A$.

Shape, basis, interpolation functions.

Property of finite-dimensional space.



$1 \leq e \leq n_{el}$

$\eta = \{1, 2, \cdots, \underbrace{n_{np}}_{\#\text{ of nodal pts.}}\}$ Set of global node numbers

η_g : Subset of nodes on Γ_g.

$\eta - \eta_g$: complement of η_g in η.

$\dim(\eta - \eta_g) = \underbrace{n_{eq}}_{\#\text{ of eqn.}}$.

N_A is the shape fun associated with node A.

$$\mathcal{V}^h \ni w^h = \sum_{A \in \eta - \eta_g} C_A N_A.$$

$$\mathcal{V}^h \ni v^h = \sum_{B \in \eta - \eta_g} d_B N_B.$$

Now, $$\mathcal{V}^h = \Big\{ w^h : w^h = \sum_{A \in \eta - \eta_g} \overset{\overset{\mathbb{R}}{\downarrow}}{C_A} N_A \Big\}$$

$$\mathcal{S}^h = \Big\{ u^h : u^h = v^h + g^h, \ v^h = \sum_{B \in \eta - \eta_g} d_B N_B \Big\}$$

$$a(w^h, u^h) \overset{\overset{\text{bilinearity}}{\downarrow}}{=} a(w^h, v^h) + a(w^h, g^h)$$

$$(G) : \quad a(w^h, v^h) = \underbrace{(w^h, f) + (w^h, h)_{\Gamma_h} - a(w^h, g^h)}$$

the given data are collected in RHS.

$$\Rightarrow \sum_{B\in\eta-\eta_g} a(N_A, N_B)\, d_B = (N_A, f) + (N_A, h)_{\Gamma_h} - a(N_A, g^h).$$

for $A \in \eta - \eta_g$.

Destination

Integer

ID Array : $$ID(A) = \begin{cases} P & \text{if } A \in \eta - \eta_g \\ 0 & \text{if } A \in \eta_g \end{cases}$$

$1 \le A \le n_{np}$ $\qquad$ $1 \le P \le n_{eq}$

$$K_{PQ} = a(N_A, N_B) \qquad P = ID(A) \qquad Q = ID(B)$$

$$d_Q = d_B$$

$$F_P = (N_A, f) + (N_A, h)_{\Gamma_h} - a(N_A, g^h).$$

$$K = [K_{PQ}] \qquad d = [d_Q] \qquad F = [F_P]$$

$$\underline{(M) \qquad Kd = F}$$

the matrix problem.

Due to its historical origin in the analysis of structures, we call

K : stiffness matrix

d : disp. vector

F : force vector.

and our finally obtained temperature field is

$$u^h = \sum_{B\in\eta-\eta_g} d_B N_B + g^h \longleftarrow \text{Dirichlet data.}$$

Solution of $Kd=F$

Remark 1: The choice of g^h is <u>not</u> unique. In practice, we often choose

$$g^h = \sum_{B\in\eta_g} N_B(x)\, g_B,$$

nodal interpolation of g data.

Remark 2: There are two primary approximations :

- the ~~solution~~ function space $\mathcal{V}^h \subset \mathcal{V}$ $\quad \mathcal{S}^h \subset \mathcal{S}$
- the mesh $\bigcup \Omega^e$ is an approximation of Ω.

Also, g^h is, in general, an approximation of the g data.

Thus, $(S) \approx (G)$.

Remark 3. If the integrals are ~~performed~~ exactly calculated,

$$K_{PQ} = a(N_A, N_B). \quad \text{meaning}$$

$$(G) \Leftrightarrow (M).$$

See Hughes book p. 191.

So we have

$$\boxed{(S) \overset{\text{Smoothness}}{\Leftrightarrow} (W) \simeq (G) \overset{\text{quadrature}}{\Leftrightarrow} (M)} \quad \text{linear solver.}$$

We are concerned with the quality of this approximation : error estimate.

Main properties of K matrix :

Theorem 1 : K is symmetric & positive definite.

proof. i) $K_{PQ} = a(N_A, N_B) \underset{\substack{\uparrow \\ a(\cdot,\cdot) \text{ is symmetric.}}}{=} a(N_B, N_a) = K_{QP}.$

ii) $$c^T K c = \sum_{P,Q=1}^{n_{eq}} c_P K_{PQ} c_Q = \sum_{A,B \in \eta - \eta_g} \bar{c}_A \, a(N_A, N_B) \, \bar{c}_B.$$

$$= a\Big(\sum_{A \in \eta - \eta_g} \bar{c}_A N_A , \sum_{B \in \eta - \eta_g} \bar{c}_B N_B \Big)$$

$$= a(w^h, w^h) \geq 0$$

$c = \{ c_P \}$

$\Updownarrow$

$w^h = \sum_{A \in \eta - \eta_g} \bar{c}_A N_A$

$c_P = \bar{c}_A$

$$C^T K C = 0 \Leftrightarrow a(w^h, w^h) = 0 \Leftrightarrow w^h_{,i} \kappa_{ij} w^h_{,j} = 0$$

$\Leftrightarrow w^h_{,i} = 0$ then w^h is a constant. Since $w^h = 0$ on Γ_g, $w^h = 0$ and consequently $C = 0$. ▨

Remark 1: Symmetry is beneficial for the data structure in implementation;

Positive definiteness ensures K is invertible.

Both are fundamentally from the properties of (W).

Remark 2: If N_A's are compact supported, K is sparse.

The global point of view: a 1D example.

$\Omega = (0, 1) \qquad \Gamma_g = \{1\} \qquad \Gamma_h = \{0\} \qquad \kappa = 1.$

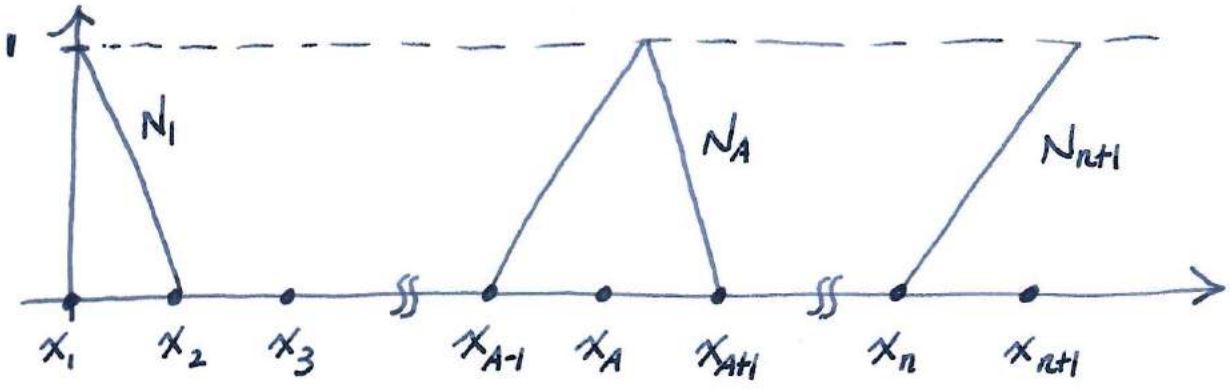
$$\Rightarrow \quad a(w, u) = \int_0^1 w_{,x} u_{,x} \, dx$$

$$(w, f) = \int_0^1 w f \, dx$$

$$(w, h)_{\Gamma_h} = w(0) h.$$

$$\mathcal{S} = \{ u : u \in H^1, \ u(1) = g \}$$

$$\mathcal{V} = \{ w : w \in H^1, \ w(1) = 0 \}$$



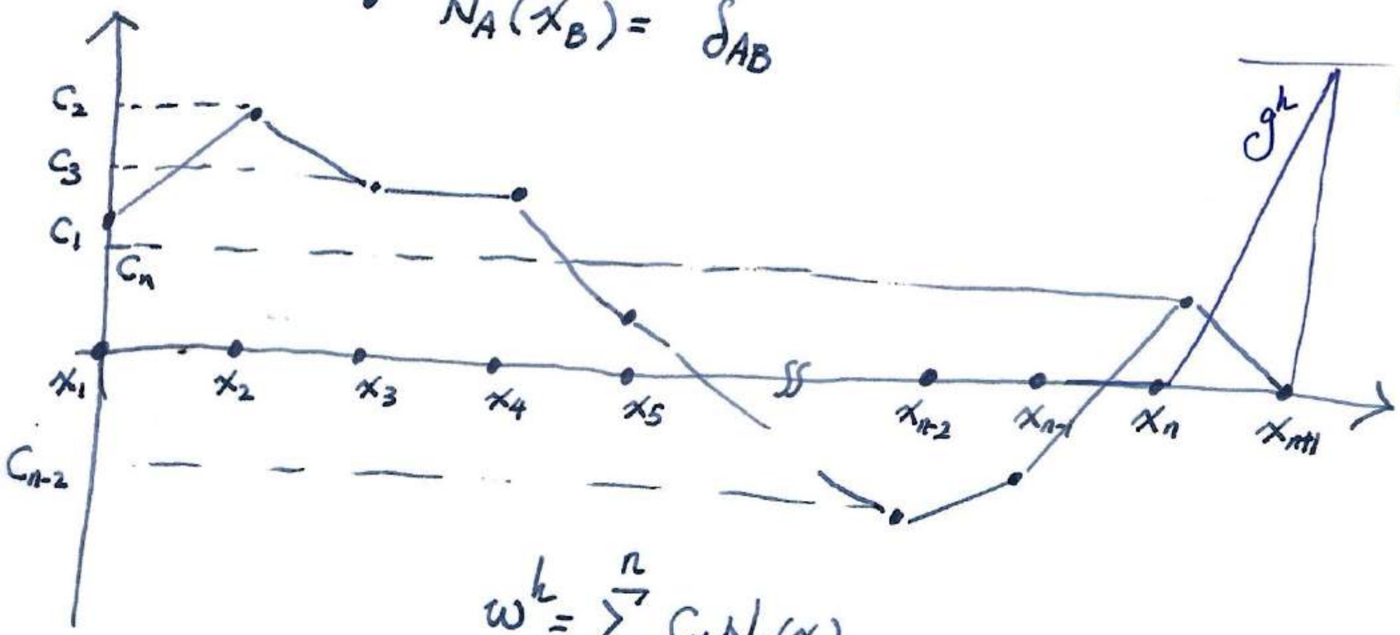
$$N_A = \begin{cases} \frac{x - x_{A-1}}{h_{A-1}} & x_{A-1} \le x \le x_A \\ \frac{x_{A+1} - x}{h_A} & x_A \le x \le x_{A+1} \\ 0 & \text{otherwise} \end{cases}$$

global viewpoint - functions are defined globally.

$h_A = x_{A+1} - x_A$

mesh parameter $h = \max h_A$.

- hat / roof / chapeau functions.
- $N_A(x_B) = \delta_{AB}$



$$w^h = \sum_{A=1}^{n} c_A N_A(x)$$

$$\eta = \{1, 2, \dots, n+1\}$$

$$\eta_g = \{n+1\}$$

$$N_{A,x} = \begin{cases} \frac{1}{h_{A-1}} & x_{A-1} \leq x \leq x_A \\ -\frac{1}{h_A} & x_A \leq x \leq x_{A+1} \\ 0 & \text{otherwise.} \end{cases}$$

$$K_{PQ} = a(N_A, N_B) = \int_0^1 N_{A,x} N_{B,x} \, dx$$

$$B = A-1 : \quad \int_0^1 N_{A,x} N_{A-1,x} \, dx = \int_{x_{A-1}}^{x_A} \frac{1}{h_{A-1}} \frac{-1}{h_{A-1}} \, dx = -\frac{1}{h_{A-1}}$$

$$B = A : \quad \int_0^1 N_{A,x} N_{A,x} = \int_{x_{A-1}}^{x_A} \frac{1}{h_{A-1}^2} \, dx + \int_{x_A}^{x_{A+1}} \frac{1}{h_A^2} \, dx = \frac{1}{h_{A-1}} + \frac{1}{h_A}$$

$$B = A+1 : \quad \int_0^1 N_{A,x} N_{A+1,x} \, dx = \int_{x_A}^{x_{A+1}} \frac{-1}{h_A} \frac{1}{h_A} \, dx = -\frac{1}{h_A}.$$

$$K = \begin{bmatrix} \frac{1}{h_1} & \frac{-1}{h_1} & . & & \text{Zeros} & \\ -\frac{1}{h_1} & \frac{1}{h_1} + \frac{1}{h_2} & \frac{-1}{h_2} & & & \\ & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{h_{A-1}} & \frac{1}{h_{A-1}} + \frac{1}{h_A} & -\frac{1}{h_A} & \\ \text{Zeros} & & & \ddots & \ddots & \ddots \\ & & & & \frac{-1}{h_{n-1}} & \frac{1}{h_{n-1}} \end{bmatrix}_{n \times n}.$$

not only sparse but also banded with 3 non-Zero entries per row.