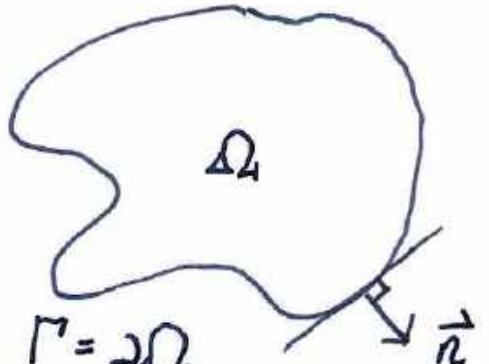
Review of linear problems

1. Heat conduction — A scalar problem posed in 1/2/3D

$$1/2/3\text{D} = n_{sd} \quad \text{\# of spatial dim.}$$



$\Gamma = \partial\Omega$ is smooth.

$n_{sd} = 2$. $\quad 1 \leq i, j, k, l \leq n_{sd}$

$$x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} = \{x_i\}$$

$$n = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \{n_i\}$$

temperature $u : \bar{\Omega} \to \mathbb{R}$ $\qquad \bar{\Omega} = \Omega \cup \Gamma$

heat flux $\vec{q} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$ $\qquad \kappa = [\kappa_{ij}] = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix}$

Gen. Fourier's law: $q_i = -\underbrace{\kappa_{ij}\, u_{,j}}_{\text{Summation convention.}}$

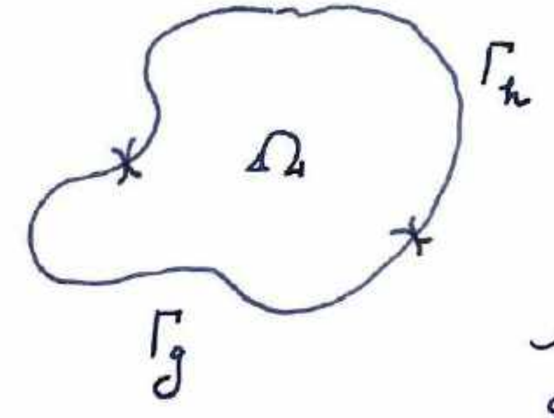
- κ is symmetric, i.e., $\kappa_{ij} = \kappa_{ji}$
- κ is positive definite, i.e., $\vec{c}^T \kappa \vec{c} \geq 0$ for all vectors $\vec{c} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$

  $\vec{c}^T \kappa \vec{c} = 0$ implies $\vec{c} = \vec{0}$

- $\kappa = \kappa(\vec{x})$ is inhomogeneous / heterogeneous.

otherwise, κ is homogeneous.

- κ is isotropic if $\kappa = \underset{\substack{\downarrow \\ \text{scalar}}}{\kappa}[\delta_{ij}] = \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix}$



$\Gamma_h \cap \Gamma_g = \emptyset$ $\overline{\Gamma_h \cup \Gamma_g} = \Gamma$

non-overlaping subdivision of Γ.

$f : \Omega \to \mathbb{R}$ heat supply per unit volume

$g : \Gamma_g \to \mathbb{R}$ prescribed boundary temp.

$h : \Gamma_h \to \mathbb{R}$ " " heat flux.

Strong form of the boundary-value problem

$$
(S) \begin{cases}
\text{Given } f : \Omega \to \mathbb{R}, \ h : \Gamma_h \to \mathbb{R}, \ g : \Gamma_g \to \mathbb{R} \\
\quad \kappa : \Omega \to \mathbb{R}^{n_{sd} \times n_{sd}}, \ \text{find } u : \bar{\Omega} \to \mathbb{R} \text{ such that} \\
\nabla \cdot \vec{q} = q_{i,i} = -\left(\kappa_{ij} u_{,j}\right)_{,i} = f \quad \text{in } \Omega \\
\qquad\qquad u = g \quad \text{on } \Gamma_g \\
-\vec{q} \cdot \vec{n} = -q_i n_i = \kappa_{ij} u_{,j} n_i = h \quad \text{on } \Gamma_h.
\end{cases}
$$

Neumann boundary condition.

Dirichlet boundary condition

Remark 1. We present the math problem with heat conduction as the background. Yet, the problem is rather general, and we call it the "elliptic boundary-value problem."

For example.

		u	$\vec{q}$	κ	f	$q=-\kappa\nabla u$
heat conduction	↔ energy conservation :	temp.	heat flux	heat conductivity	heat source	Fourier's law
deformation of an elastic bar	↔ linear momentum conservation :	disp	stress	Young's modulus	body force	Hooke's law

See. Baker, Carey, Oden, P. 44.

Remark 2: There are other type of boundary conditions, e.g.,

$$\lambda u - q_i n_i = h.$$

or.

$$-q_i n_i = -\beta(u - u_{ref}).$$

is known as the Robin boundary condition.

Remark 3: For a precise statement, we need to provide the spaces to which the functions belong. See Hughes book Appendix 1.1 & Arbogast Bona Chap. 8.

Some function spaces :

$$L_2 \doteq L_2(\Omega) = \left\{ \omega : \int_0^1 \omega^2 \, dx < \infty \right\}$$

$$H^k = H^k(\Omega) = \left\{ \omega : \omega \in L_2 \quad \omega_{,i} \in L_2 , \cdots , \omega_{,\underbrace{ij\cdots l}_{k \text{ indices.}}} \in L_2 \right\}$$

(e.g. $n_{sd} = 2$. $$H^2(\Omega) = \left\{ \omega : \omega \in L_2 .\ \omega_{,x} \in L_2 \quad \omega_{,y} \in L_2 \quad \omega_{,xx} \in L_2 \quad \omega_{,xy} \in L_2 \quad \omega_{,yy} \in L_2 \right\}$$

- $L_2 = H^0$ apparently

- We denote $H_0^1(\Omega) = \left\{ \omega : \omega \in H^1(\Omega) \quad \omega = 0 \text{ on } \partial\Omega \right\}$

Divergence theorem :

$$\int_\Omega \nabla \cdot \vec{q} \, d\Omega = \int_{\partial\Omega} \vec{q} \cdot \vec{n} \, d\Gamma .$$

or $$\int_\Omega q_{i,i} \, d\Omega = \int_{\partial\Omega} q_i n_i \, d\Gamma$$

Integration-by-parts :

$$\int_\Omega f \, \nabla \cdot \vec{g} \, d\Omega = - \int_\Omega \nabla f \cdot \vec{g} + \int_{\partial\Omega} f \vec{g} \cdot \vec{n} \, d\Gamma$$

or $$\int_\Omega f \, g_{i,i} \, d\Omega = - \int_\Omega f_{,i} \, g_i \, d\Omega + \int_{\partial\Omega} f g_i n_i \, d\Gamma$$

$\mathcal{S} := \{ u : u \in H^1(\Omega),\ u|_{\Gamma_g} = g \}$ is the trial solution space

$\mathcal{V} := \{ w : w \in H^1(\Omega),\ w|_{\Gamma_g} = 0 \}$ is the test/weighting function space.

Weak or variational form of the BV problem

(W)

Given $f : \Omega \to \mathbb{R}$, $h : \Gamma_h \to \mathbb{R}$, $g : \Gamma_g \to \mathbb{R}$, and κ ~~θ~~ $: \Omega \to \mathbb{R}^{n_{sd} \times n_{sd}}$, find $u \in \mathcal{S}$ such that for all $w \in \mathcal{V}$

weak or generalized solution

$$a(w, u) = (w, f) + (w, h)_{\Gamma_h}$$

variational eqn. or equation of virtual work.

where

$$a(w, u) = \int_\Omega w_{,i}\, \kappa_{ij}\, u_{,j}\, d\Omega$$

$$(w, f) = \int_\Omega w f \, d\Omega$$

$$(w.\, h)_{\Gamma_h} = \int_{\Gamma_h} w.\, h \, d\Gamma.$$

also called virtual disp. in mechanics

The equivalence of (S) and (W)

Proposition a: let u be a solution of (S), then u is a solution of (W).

Proof: u is a solution of (S), we have

$$~~\#~~ (\kappa_{ij} u_{,j})_{,i} + f = 0$$

$$\Rightarrow \int_\Omega w(\kappa_{ij} u_{,j})_{,i} + w f \overset{d\Omega}{=} 0 \quad \text{for } \forall w \in \mathcal{V}.$$

= integration-by-parts ⇒

$$-\int_{\Omega} w_{,i}\, \kappa_{ij}\, u_{,j}\, d\Omega + \int_{\partial\Omega} w\, \kappa_{ij}\, u_{,j}\, n_i\, d\Gamma$$

$$+ \int_{\Omega} w f\, d\Omega = 0$$

$$\int_{\partial\Omega} w\, \kappa_{ij}\, u_{,j}\, n_i\, d\Gamma = \int_{\Gamma_h} w\, h\, d\Gamma$$ because $w \in \mathcal{V}$ and u satisfies the Neumann B.C.

$$\Rightarrow \quad a(w, u) = (w, f) + (w, h)_{\Gamma_h}.$$

Thus. u solves (W).

▨

proposition b: let u be a solution of $(\overset{W}{\cancel{S}})$ (and u is smooth enough for ~~th~~ second derivatives), then u is a solution of (S).

Proof: Let u be a solution of (W),

$$\int_{\Omega} w_{,i}\, \kappa_{ij}\, u_{,j}\, d\Omega = \int_{\Omega} w f\, d\Omega + \int_{\Gamma_h} w\, h\, d\Gamma.$$

$\|$ ← use of int.-by-parts & we assume $u_{,ji}$ exists or u is twice differentiable

$$-\int_{\Omega} w\, (\kappa_{ij}\, u_{,j})_{,i}\, d\Omega + \int_{\underset{\partial\Omega}{\overset{\Gamma}{\|}}} w\, \kappa_{ij}\, u_{,j}\, n_i\, d\Gamma.$$

$$\Rightarrow \int_{\Omega} w \left[(\kappa_{ij}\, u_{,j})_{,i} + f \right] d\Omega + \int_{\Gamma_h} w \left[h - \kappa_{ij}\, u_{,j}\, n_i \right] d\Gamma = 0.$$

↑ Euler-Lagrange eqn of the weak / variational formulation.

Now, Since $u \in \mathcal{S}$, $u = g$ on Γ_g is satisfied already due to the construction of the trial solution space. We only need to show $(\kappa_{ij} u_{,j})_{,i} + f = 0$ in Ω &

$$(\kappa_{ij} u_{,j}) n_i - h = 0 \quad \text{on } \Gamma_h .$$

We can choose $\widehat{w} = \phi \left((\kappa_{ij} u_{,j})_{,i} + f \right)$ with $\phi > 0$ in Ω and $\phi = 0$ on $\Gamma = \partial\Omega$. Of course, $\widehat{w} \in \mathcal{V}$, and we may insert $\hat{w}$ into E.-L. :

$$\int_{\Omega} \phi \left[(\kappa_{ij} u_{,j})_{,i} + f \right]^2 d\Omega = 0$$

$$\Rightarrow (\kappa_{ij} u_{,j})_{,i} + f = 0. \quad \text{in } \Omega.$$

Next, choosing $\hat{w} = \psi (h - \kappa_{ij} u_{,j} n_i)$ with $\psi > 0$ on $\partial\Omega$, we can establish $(\kappa_{ij} u_{,j}) n_i = h$ on Γ_h.

■

Remark 1 : We established $(S) \Longleftrightarrow (W)$, under the assumptio that the weak solution is twice differentiable.

Remark 2 : The Dirichlet B.C. is built into the def. of $\mathcal{S}$, and B.C. of this type is called essential boundary conditions ;

the Neumann B.C. is built implicitly in the variational eqn, and B.C. of this type is called natural boundary conditions.

Remark 3: The technique used after the E.-L. eqn is known as the fundamental lemma of the calculus of variations, which transform a weak form to its corresponding Strong form. With this procedure, one may get the mathematical features of a weak form through the Euler-Lagrange equations.

Remark 4: $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ are symmetric bilinear forms.

Symmetry: $a(w,u) = a(u,w)$

$(w,u) = (u,w)$.

Bilinearity: linearity in both slots

$$a(c_1 w_1 + c_2 w_2, u) = c_1 a(w_1, u) + c_2 a(w_2, u)$$

$$a(w, c_1 u_1 + c_2 u_2) = c_1 a(w, u_1) + c_2 a(w, u_2).$$

Ref. Hughes. FEM book. Sec 1.1 – 1.4, Sec 2.1 – 2.3

Physical Problem	Conservation Principle	State Variable, u	Flux, σ	Material Modulus, k	Source, f	Constitutive Equation, $\sigma = -ku'$
Deformation of an elastic bar	Equilibrium of forces (conservation of linear momentum)	Displacement	Stress	Young's modulus of elasticity	Body forces	Hooke's law
Heat conduction in a rod	Conservation of energy	Temperature	Heat flux	Thermal conductivity	Heat sources	Fourier's law
Fluid flow	Conservation of linear momentum	Velocity	Shear stress	Viscosity	Body forces	Stokes' law
Electrostatics	Conservation of electric flux	Electric potential	Electric flux	Dialectric permittivity	Charge	Coulomb's law
Flow through porous media	Conservation of mass	Hydraulic head	Flow rate	Permeability	Fluid source	Darcy's law

FIGURE 2.1 *Interpretation of physical variables and equations for various types of physical problems.*