Homework 5
Due: June 10, 2024

1. Consider the strain energy $\Psi(\boldsymbol{E})$, which can be represented in terms of $\boldsymbol{F}$, i.e. $\Psi(\boldsymbol{E}) = \widehat{\Psi}(\boldsymbol{F})$. Let

$$\mathbb{A}_{iIjJ} := \frac{\partial^2 \widehat{\Psi}}{\partial F_{iI} \partial F_{jJ}}.$$

Show that $\mathbb{A}_{iIjJ} = \delta_{ij} S_{IJ} + F_{iK} F_{jL} \mathbb{C}_{IKJL}$.

2. Design a new implementation of the generalize-α scheme using the following options:

(a) the predictor is given by a zero acceleration, i.e., $a_{n+1}^{(0)} = 0$;

(b) in the multi-corrector, the incremental variable is $\Delta d_{n+1}^{(i+1)}$.

State your algorithm implementation in steps.

3. Consider the following system:

$$\boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{0}$$

$$\boldsymbol{d} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}.$$

Assume $k_1 = 10^4, k_2 = 1, m_1 = 1,$ and $m_2 = 1$.

The intent of this two-degree-of-freedom model is to represent the character of typical large systems. The first mode is intended to represent those modes that are physically important and must be accurately integrated. The second mode represents the spurious high frequencies. It is desirable that the step-by-step integrator filter these high modes from the response of the system.

a. Determine the natural frequencies ω_1 and ω_2 of this system.

b. Consider the initial-value problem for the system above with initial data given by

$$\boldsymbol{d_0} = \begin{Bmatrix} 1 \\ 10 \end{Bmatrix}, \quad \boldsymbol{v_0} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The initial condition is designed to exacerbate overshoot phenomena.

Write a computer program to solve this problem employing the following methods.

1. Central difference method, i.e., Newmark method with $\beta = 0$ and $\gamma = 1/2$.
2. Trapezoidal rule, i.e., Newmark method with $\beta = 1/4$ and $\gamma = 1/2$.
3. Damped Newmark method $(\beta = 0.3025, \gamma = 0.6)$
4. Generalized-α method with $\rho_\infty = 0$, 0.5, 1.0, respectively. Notice that the parameterization is given by $\alpha_m = \frac{2-\rho_\infty}{1+\rho_\infty}$, and $\alpha_f = \frac{1}{1+\rho_\infty}$. The rest two parameters are chosen to guarantee second-order accuracy.

Run the program at $\Delta t = T_1/20$ over a time interval of $[0, 5T_1]$. Obtain time-history plots for the displacements and velocities in each case. Comment on the relative effectiveness of the algorithms.