Homework 3
Due: Apr. 15 2023

1. Starting with the strong form for the 1D nonlinear heat conduction problem,

$$(S)\begin{cases} q_{,x} = f & \text{on } \Omega =]0,1[\\ u(1) = g & \text{on } \Gamma_g \\ (-1)(-q(0)) = h & \text{on } \Gamma_h \\ q(u, u_{,x}; x) = -\kappa(u; x) u_{,x} & \end{cases},$$

the element arrays of the matrix problem (M) are

$$n_a^e(\mathbf{d^e}) = -\int_{\Omega^e} N_{a,x} q \, dx \ ,$$

$$f_a^e = \int_{\Omega^e} N_a f \, dx + \begin{cases} h\delta_{a1} & e = 1 \\ 0 & e > 1 \end{cases},$$

$$\frac{\partial n_a^e}{\partial d_b^e} = \left(\int_{\Omega^e} N_{a,x} \frac{\partial \kappa(u^h;x)}{\partial u} N_b \left(\sum_{c=1}^{n_{en}} N_{c,x} \, d_c^e \right) dx \right) + \int_{\Omega^e} N_{a,x} \kappa(u^h; x) N_{b,x} \, dx \ .$$

a. Derive the arrays

$$n^e(d^e) = \{n_a^e(d^e)\}, \qquad f^e = \{f_a^e\}, \qquad Dn^e(d^e) = \left[\frac{\partial n_a^e}{\partial d_b^e}\right]$$

for three node quadratic finite element using the 2-point Gaussian rule.

b. Let $\kappa(u; x) = 1 + u^2$, modify the FEM-1D-demo code (https://github.com/M3C-Lab/FEM-tutorial/tree/main/Chapter-nonlinear) to solve the problem with consistent Newton-Raphson method. Report the convergence rate of the error with linear, quadratic, and cubic elements.

2. Consider $\boldsymbol{N}(\boldsymbol{d}) = \boldsymbol{F}^{ext}$, where

$$\boldsymbol{N}(\boldsymbol{d}) = \begin{Bmatrix} N_1(\boldsymbol{d}) \\ N_2(\boldsymbol{d}) \end{Bmatrix} = \begin{Bmatrix} N_1(d_1, d_2) \\ N_2(d_1, d_2) \end{Bmatrix},$$

$$N_1(d_1, d_2) = \frac{x d_1}{10 - d_1} - 0.5 d_2^2,$$

$$N_2(d_1, d_2) = d_2 - d_1,$$

$$\boldsymbol{F}^{ext} = \begin{Bmatrix} F_1^{ext} \\ F_2^{ext} \end{Bmatrix},$$

$$F_2^{ext} = 0$$

Incremental load steps:

$$F_1^{ext} = F_i \qquad \begin{aligned} F_0 &= 0 \\ F_1 &= 0.25 \\ F_2 &= 0.5 \\ &\vdots \\ F_{40} &= 10.0 \end{aligned}$$

For both $x = 15$ and $x = 25$ solve for $\boldsymbol{d}$ using the following:

i) Newton-Raphson (consistent tangent)

ii) Newton-Raphson (consistent tangent) with line search

iii) Modified N-R (consistent tangent on first iteration in each step)

iv) Repeat iii) with line search (maximum 5 iterations in for search parameter)
Accept s when $|G(s)| \leq 0.5|G(0)|$. If no acceptance s then exit.
v) Modified N-R with BFGS
vi) Modified N-R with line search and BFGS

Convergence test: $\|\mathbf{R}\| = \sqrt{R_1^2 + R_2^2}$, $\left\|\mathbf{R}_{\mathrm{n+1}}^{(\mathrm{i+1})}\right\| \leq \varepsilon\|\mathbf{R}_{\mathrm{n+1}}^{0}\|, \varepsilon = 10^{-4}$.

In all cases limit the maximum number of iterations to 15.

Plot:

a. Exact $N_1(d_1, d_2 = d_1)$ vs. d_1 for $x = 15$ and $x = 25$
b. Numerical $N_1(d_1, d_2 = d_1)$ vs. d_1 for all cases
c. Number of iterations to convergence vs. load step number for all cases
d. Consider the 1D heat equation. Modify the code and experiment it with the nonlinear heat conduction.

3. Consider an alternate definition of the BFGS vectors defined by the line search function $G(s)$. Recall that

$$G^{(i)}\left(s^{(i)}\right) := \Delta d^{(i)} \cdot R\left(d^{(i)} + s^{(i)} \Delta d^{(i)}\right),$$

and subtract

$$G^{(i)}(0) := \Delta d^{(i)} \cdot R(d^{(i)})$$

to get the identity

$$G^{(i)}\left(s^{(i)}\right) - G^{(i)}(0) = \Delta d^{(i)} \cdot \Delta R^{(i)}.$$

Now we define the BFGS vectors as

$$v^{(i)} := \frac{\Delta d^{(i)}}{G^{(i)}(s^{(i)}) - G^{(i)}(0)},$$

$$w^{(i)} = -\Delta R^{(i)} + \alpha^{(i)} R^{(i)},$$

$$\alpha^{(i)} := \sqrt{\frac{-s^{(i)}\left(G^{(i)}(s^{(i)}) - G^{(i)}(0)\right)}{G^{(i)}(0)}}.$$

Verify that the quasi-Newton equation is satisfied.