Homework 8

Due: June 5, 2023

1. Considering the evolution equations of variables $\boldsymbol{Q}^{\alpha}$ given by

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{Q}^{\alpha} + \dfrac{1}{\tau^{\alpha}}\boldsymbol{Q}^{\alpha} = \dfrac{d}{dt}\tilde{\boldsymbol{S}}^{\alpha}_{\text{iso}}\,, \\ \boldsymbol{Q}^{\alpha}|_{t=0} = \boldsymbol{Q}^{\alpha}_{0}\,, \end{cases} \qquad \alpha = 1, \dots, m.$$

a. Derive the closed form solutions for the above equations, which take the form of hereditary integral

$$\boldsymbol{Q}^{\alpha} = \exp\left(-\frac{t}{\tau^{\alpha}}\right)\boldsymbol{Q}^{\alpha}_{0} + \int_{0^{+}}^{t} \exp\left(-\frac{t-s}{\tau^{\alpha}}\right)\frac{d}{ds}\tilde{\boldsymbol{S}}^{\alpha}_{\text{iso}}\,ds.$$

b. Obtain the following one-step, unconditionally stable and second-order accuracy recurrence update formula for $\boldsymbol{Q}^{\alpha}_{n+1}$ as

$$\boldsymbol{Q}^{\alpha}_{n+1} = \exp(\xi^{\alpha})\tilde{\boldsymbol{S}}^{\alpha}_{\text{iso n+1}} + \exp(\xi^{\alpha})(\exp(\xi^{\alpha})\boldsymbol{Q}^{\alpha}_{n} - \tilde{\boldsymbol{S}}^{\alpha}_{\text{iso n}}),$$

where $\xi^{\alpha} := -\Delta t_{n}/2\tau^{\alpha}$, and τ^{α} is the relaxation time for the α-th process.

c. In the derivation of the above recurrence update formula, a mid-point rule is applied for the exponential term in the time interval. There exists an alternate second-order accurate recurrence update formula by applying the mid-point rule to the stress rate in the integral, see Page 355 Eq (10.3.15) in J.C. Simo and T.J.R. Hughes, "Computational Inelasticity", Springer Science & Business Media, 2006. Invoke that strategy and obtain the recurrence formula

$$\boldsymbol{Q}^{\alpha}_{n+1} = \exp(2\xi^{\alpha})\boldsymbol{Q}^{\alpha}_{n} - \frac{1-\exp(2\xi^{\alpha})}{2\xi^{\alpha}}(\,\tilde{\boldsymbol{S}}^{\alpha}_{\text{iso n+1}} - \tilde{\boldsymbol{S}}^{\alpha}_{\text{iso n}}).$$