Homework 7
Due: May 22, 2023

1. Consider the same problem given in Problem 3 of homework 6. Modify the problem into a system of first-order ordinary differential equations as follows,

$$\boldsymbol{M}\dot{\boldsymbol{v}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{0},$$
$$\boldsymbol{v} - \dot{\boldsymbol{d}} = \boldsymbol{0}.$$

The generalized-α scheme for first order ODE problems is slightly different from the one presented in class. It can be stated as follows. At the time interval (t_n, t_{n+1}), given $\dot{\boldsymbol{v}}_n$, $\dot{\boldsymbol{d}}_n$, $\boldsymbol{v}_n$, $\boldsymbol{d}_n$, determine $\dot{\boldsymbol{v}}_{n+1}$, $\dot{\boldsymbol{d}}_{n+1}$, $\boldsymbol{v}_{n+1}$, $\boldsymbol{d}_{n+1}$, such that the following system of equations is satisfied.

$$\boldsymbol{M}\dot{\boldsymbol{v}}_{n+\alpha_m} + \boldsymbol{K}\boldsymbol{d}_{n+\alpha_f} = \boldsymbol{0},$$

$$\boldsymbol{v}_{n+\alpha_f} - \dot{\boldsymbol{d}}_{n+\alpha_m} = \boldsymbol{0},$$

$$\boldsymbol{v}_{\boldsymbol{n+1}} = \boldsymbol{v}_{\boldsymbol{n}} + (1-\gamma)\Delta t\dot{\boldsymbol{v}}_{\boldsymbol{n}} + \gamma\Delta t\dot{\boldsymbol{v}}_{\boldsymbol{n+1}},$$
$$\boldsymbol{d}_{\boldsymbol{n+1}} = \boldsymbol{d}_{\boldsymbol{n}} + (1-\gamma)\Delta t\dot{\boldsymbol{d}}_{\boldsymbol{n}} + \gamma\Delta t\dot{\boldsymbol{d}}_{\boldsymbol{n+1}},$$

and the intermediate terms are defined in the usual way, that is,

$$(\cdot)_{n+\alpha_m} = (1-\alpha_m)(\cdot)_n + \alpha_m(\cdot)_{n+1},$$

$$(\cdot)_{n+\alpha_f} = (1-\alpha_f)(\cdot)_n + \alpha_f(\cdot)_{n+1}.$$

The parameters are represented in terms of the spectral radius of the amplification matrix at the highest mode, ρ_∞, as follows,

$$\alpha_m = \frac{3-\rho_\infty}{2(1+\rho_\infty)}, \quad \alpha_f = \frac{1}{1+\rho_\infty}, \quad \gamma = \frac{1}{1+\rho_\infty}.$$

a. Give an implementation for the algorithm.
b. Using the same material parameters and initial conditions, numerically study the problem with $\rho_\infty = 0, 0.5, 1$. Compare the results with the generalized-α scheme in HW6 in terms of the velocity overshoot.

2. Consider a multiplicative modification of the stress as follows,

$$\boldsymbol{S}_{\text{alg}} = \frac{\phi(\boldsymbol{C}_{n+1}) - \phi(\boldsymbol{C}_n)}{\boldsymbol{S}_m : \boldsymbol{Z}_n}\boldsymbol{S}_m,$$

where $\boldsymbol{S}_m = \boldsymbol{S}(\boldsymbol{C}_{n+1/2})$, $\boldsymbol{Z}_{\boldsymbol{n}} = (\boldsymbol{Z}_{\boldsymbol{n+1}} - \boldsymbol{Z}_{\boldsymbol{n}})/2$. Show that this algorithmic stress satisfies the directionality property, and it thus leads to an energy-stable scheme for nonlinear elastodynamics. Different from the discrete gradient formula given in class, this one multiplies a scalar in front of $\boldsymbol{S}_m$. This algorithmic stress was originally provided in A. Chorin, et al., "Product formulas and numerical algorithms", Communications on Pure and Appliedd Mathematics, 32:205-256, 1978.

3. Consider the standard solid model. The strain can be represented in terms of the stress history in the convolution representation

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-s)\dot{\sigma}(s)ds.$$

Show that the creep function

$$J(t) = \frac{1}{E_\infty}\left(1 - \frac{E}{E_0}e^{-\frac{E_\infty}{\tau E_0}t}\right).$$